

Unit-I: propped cantilevers

- * $D_s = 1$, if reaction at prop end is treated as a Redundant force, the resulting basic determinate structure is a cantilever, if fixed end moment is the redundant, then the basic determinate structure is a Simply Supported beam.

procedures- Step 1:- Remove the prop and determine the deflection of the prop end for given loads.

Step 2:- Apply the redundant and find the deflection of the prop end without considering given loads

Step 3:- use the consistency condition for prop end to find prop reaction

Step 4:- Solve for reactions at the fixed end from equilibrium equations

Step 5:- Draw SFD and BMD.

Cantilever

1. Determinate beam
2. Equilibrium equations are sufficient for solving the beam
3. One end fix and other end free in a beam
4. POCFs are zero

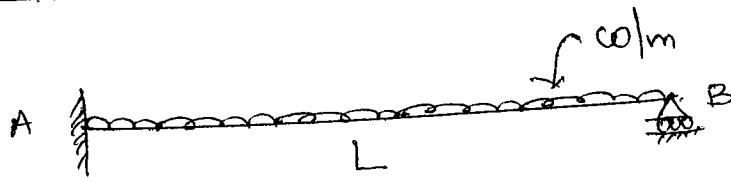
what is a propped cantilever? what is the degree of indeterminacy?

propped cantilever

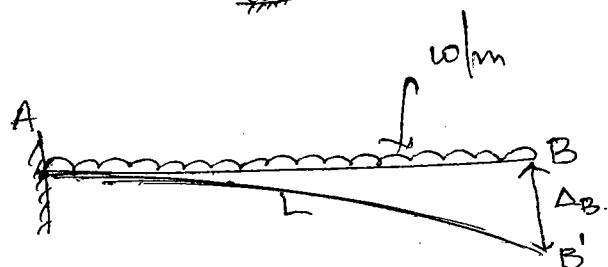
1. Indeterminate beam.
2. One additional condition called consistency cond' is required for solving
3. One end fix and other end simply supported
4. POCFs are one

Name a method for deriving the compatibility Equation for the propped cantilever?

prob:- propped cantilever subjected to UDL throughout :-

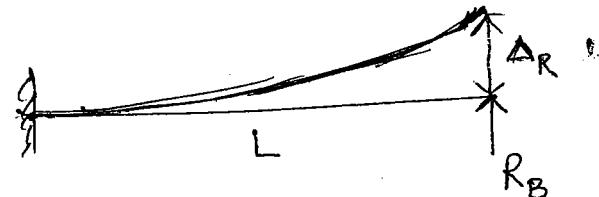


Ans:- Removing prop,



$$\Delta_{B,L} = \frac{wL^4}{8EI}$$

Applying Redundant R_B ,



$$\Delta_{B,R} = \frac{R_B L^3}{3EI} \quad (\text{-ve})$$

from Consistency condition, $\Delta_B = 0$

$$\therefore \Delta_{B,L} + \Delta_{B,R} = 0$$

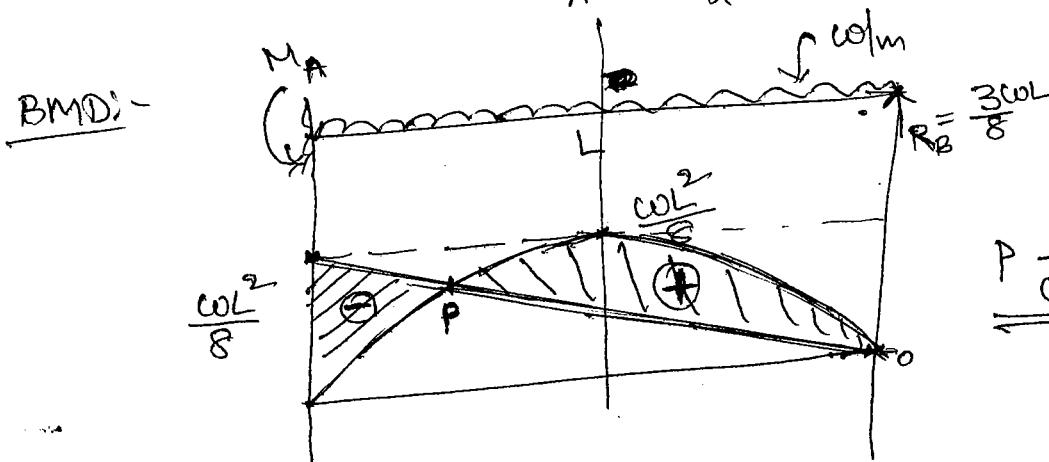
$$\therefore \frac{wL^4}{8EI} - \frac{R_B L^3}{3EI} = 0 \Rightarrow R_B = \frac{3wL}{8}$$

from eq^{lm}, $\sum F_y = 0$ $R_A + R_B = wL$

$$R_A = wL - \frac{3wL}{8} = \frac{5wL}{8} = R_A$$

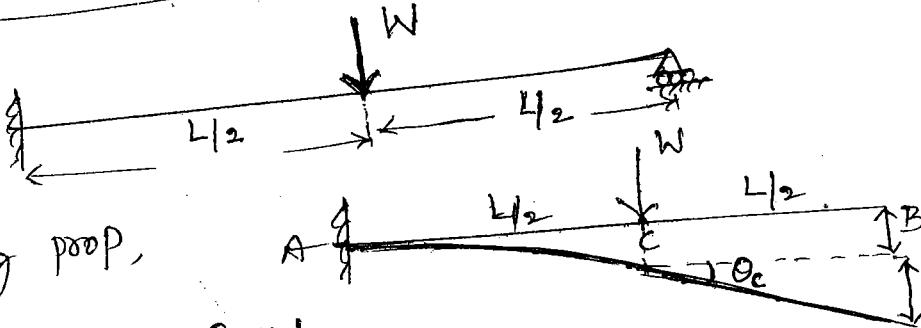
$$\therefore \sum M_A = 0 \Rightarrow M_A + R_B \times L - wL \times \frac{L}{2} = 0$$

$$M_A = \frac{wL^2}{2} - \frac{3wL^2}{8} = \frac{wL^2}{8} = M_A$$



P - Point of contraflexure

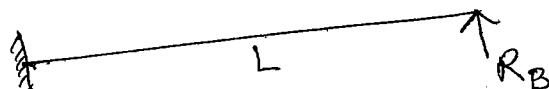
prob:- Cantilever with prop subjected to point load at centre



sol:- Removing prop,

$$\begin{aligned}\Delta_{BL} &= \Delta_c + \theta_c \times \frac{L}{2} \\ &= \frac{W(L/2)^3}{3EI} + \frac{WL(L/2)^2}{EI} \times \left(\frac{L}{2}\right) \\ &= \frac{WL^3}{24EI} + \frac{WL^3}{16EI} = \frac{5WL^3}{48EI}\end{aligned}$$

Applying Redundant-



$$\therefore \Delta_{BR} = \frac{R_B L^3}{3EI} \text{ (-ve)}$$

$$\begin{aligned}\text{from Consistency} \quad \Delta_B = 0 &\Rightarrow \Delta_{BL} + \Delta_{BR} = 0 \\ \frac{5WL^3}{48EI} - \frac{R_B L^3}{3EI} &= 0 \Rightarrow \frac{5WL^3}{48EI} = \frac{R_B L^3}{3EI}\end{aligned}$$

$$\therefore R_B = \frac{5W}{16} = \boxed{\frac{5W}{16} = R_B}$$

from Equilibrium Equations

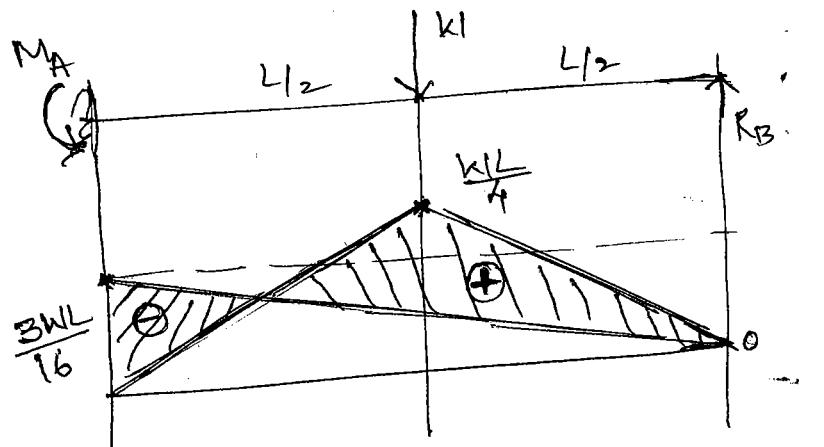
$$R_A + R_B = W \Rightarrow R_A = W - \frac{5W}{16}$$

$$\boxed{R_A = \frac{11W}{16}}$$

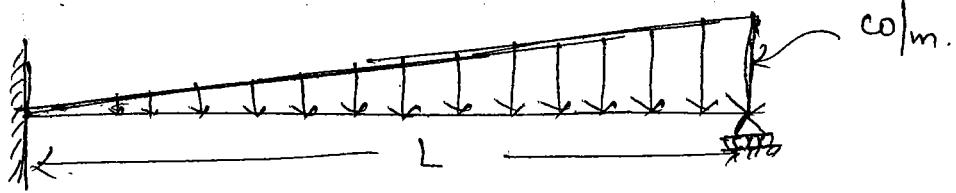
$$\stackrel{\leftarrow}{\sum M_A} = 0 \quad M_A + R_B \times L - W \times \frac{L}{2} = 0$$

$$M_A = \frac{WL}{2} - \frac{5WL}{16} = \boxed{\frac{3WL}{16} = M_A}$$

BMD:-



prob:- find the Support moment for the propped cantilever carrying UVL w/out depth from A to B. Draw BMD?



Sol:-

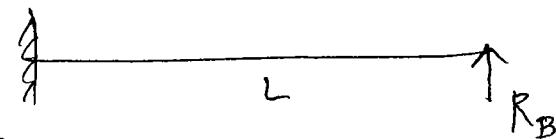
Removing the prop, A B

$$\Delta_{B,L} = \frac{11wL^4}{180EI}$$

\therefore UVL in the other way

$$\Delta_B = \frac{wL^4}{30EI}$$

Applying the Redundants,



$$\Delta_{B,R} = \frac{R_B L^3}{3EI} \quad (\text{---ve})$$

$$\Delta_{B,L} + \Delta_{B,R} = 0$$

from consistency condition, $\Delta_B = 0$

$$\therefore \frac{11wL^4}{180EI} - \frac{R_B L^3}{3EI} = 0$$

$$\therefore \boxed{\frac{11wL}{40} = R_B}$$

$$R_A + R_B = \frac{wL}{2}$$

$$R_A = \frac{wL}{2} - \frac{11wL}{40}$$

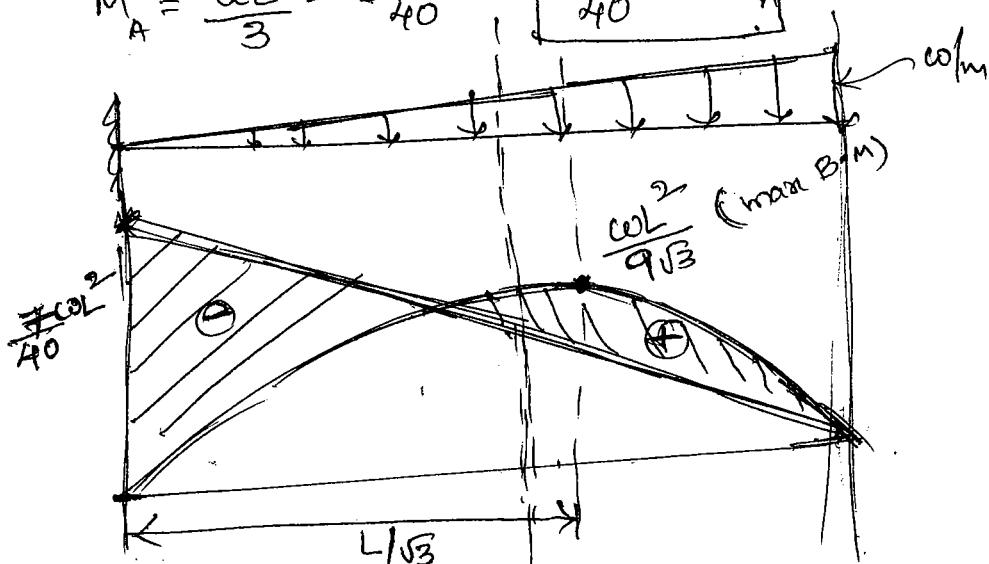
from Equilibrium Equation,

$$\text{At } A: M_A + R_B \times L - \frac{1}{2} \times L \times w \times \frac{2L}{3} = 0$$

$$M_A = \frac{wL^2}{3} - \frac{11wL^2}{40} = \boxed{\frac{7wL^2}{40} = M_A}$$

$$\boxed{R_A = \frac{9wL}{40}}$$

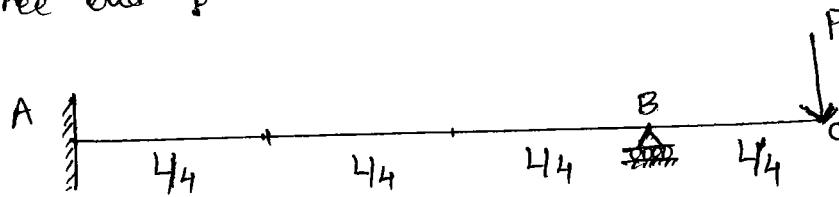
BMDN



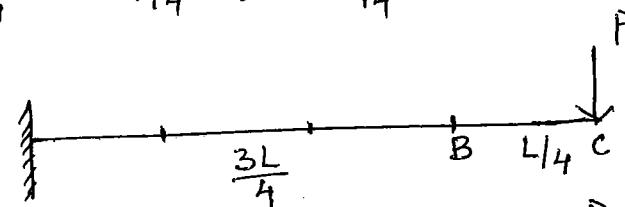
Prob1 Draw BMD for a propped cantilever with an overhang at $L/4$ from free end carrying a point load at free end.

I

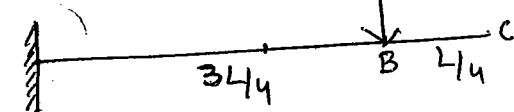
sol1



sol1- Removing the prop,



from Maxwell Reciprocal theorem,



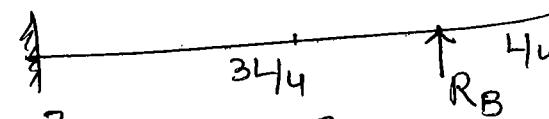
$$\Delta_B = \Delta_C$$

$$\therefore \text{finding } \Delta_C = \Delta_B + O_B \times \frac{L}{4} = \frac{P(\frac{3L}{4})^3}{3EI} + \frac{P(\frac{3L}{4})^2}{2EI} \times \frac{L}{4}$$

$$= P \frac{\cancel{9} L^3}{64 \times 3EI} + P \frac{9L^2}{16 \times 2 \times EI} \times \frac{L}{4}$$

$$= \frac{9PL^3}{64EI} \left(1 + \frac{1}{2}\right) = \frac{9PL^3}{64EI} \times \frac{3}{2} = \frac{27PL^3}{128EI} = \Delta_{B,L}$$

Applying Redundant,



$$\Delta_{B,R} = R_B \times \frac{(\frac{3L}{4})^3}{3EI} = R_B \times \frac{9L^3}{64EI} \quad (\text{-ve})$$

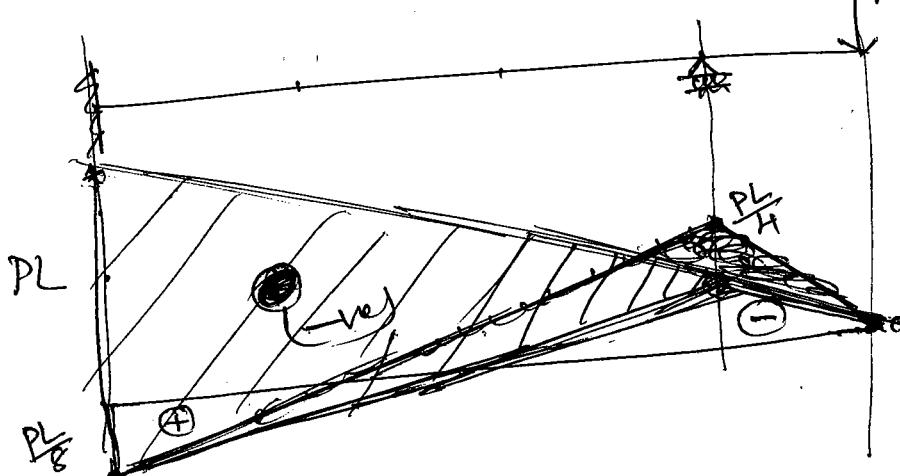
from consistency, $\Delta_B = 0 \Rightarrow \frac{27PL^3}{128EI} = R_B \times \frac{9PL^3}{64EI}$

$$\therefore R_B = \frac{3P}{2}$$

$$M_A + \frac{3P \times 3L}{4} - PL = 0$$

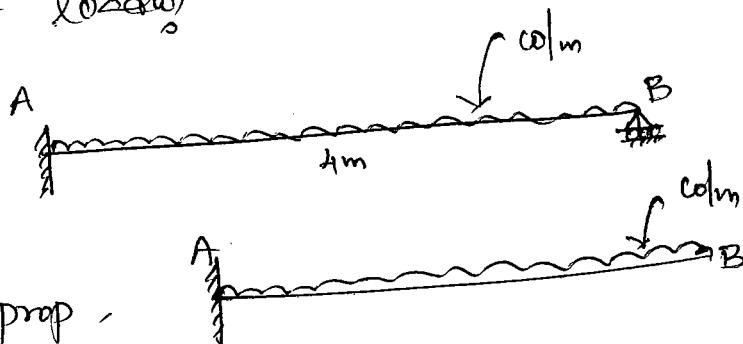
$$M_A = PL - \frac{9PL}{8} = -\frac{PL}{8} \quad (\text{C.W})$$

BMD1-



Prob:- A timber beam $12 \times 20\text{cm}$ and 4m long is loaded with a UDL. It is fixed at left end and simply supported at right end. If maximum allowable fibre stress is 10N/mm^2 and right support settles by an amount equal to $\frac{\text{col}^4}{24EI}$, determine permissible value of loading.

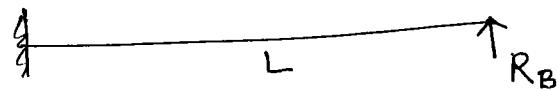
Sol:-



Removing prop -

$$\Delta_{B,L} = \frac{COL^4}{8EI}$$

Applying Redundant,



$$\Delta_{B,R} = \frac{R_B L^3}{3EI} \quad (\text{-ve})$$

Consistency $\Delta_B = \Delta_{B,L} - \Delta_{B,R} = \frac{COL^4}{24EI}$

$$\Rightarrow \frac{COL^4}{8EI} - \frac{R_B L^3}{3EI} = \frac{COL^4}{24EI}$$

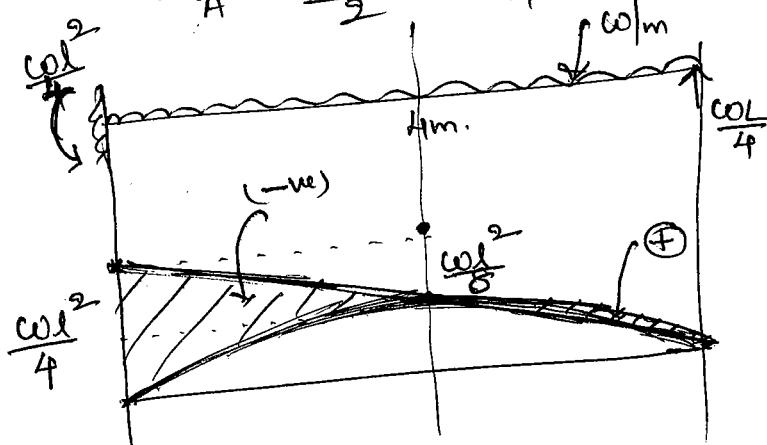
$$\frac{COL^4}{8EI} \left(1 - \frac{1}{3}\right) = \frac{R_B L^3}{3EI}$$

$$\frac{COL^4}{48EI} \times \frac{2}{3} = \frac{R_B \times L^3}{3EI} \Rightarrow R_B = \frac{WL}{4}$$

$\therefore \sum M_A = 0$ $M_A + R_B \times L - w \times l \times \frac{l}{2} = 0$

$$M_A = \frac{COL^2}{2} - \frac{COL \times l}{4} = \frac{COL^2}{2} \left(1 - \frac{1}{2}\right) = \frac{COL^2}{4}$$

BMD



$$M_{\text{max}} = \frac{wL^4}{4}, \quad \text{Bending stress, } \frac{M}{I} = \frac{1}{y}$$

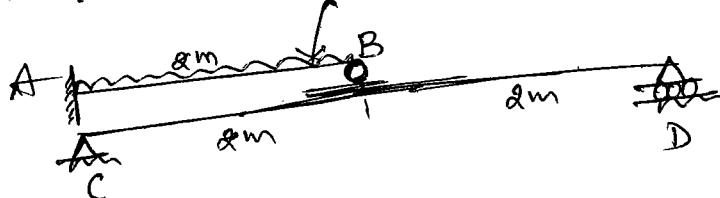
$$M = \sigma \times \frac{I}{y} \Rightarrow \frac{wL^2}{4} = 10 \times \frac{\left(\frac{bd^3}{12}\right)}{(d/2)}$$

$$w = 10 \times \frac{120 \times 200}{6} \times \frac{4}{(4000)^2}$$

$$\therefore \sigma_{\text{allowable}} = 10 \text{ N/mm}^2$$

$$w = 2 \text{ KN/m}$$

prob- A cantilever AB of length 2m is fixed at end A and B rests on SSB of span 4m at centre. Find Support moment at A and deflection at B when cantilever is loaded with UDL of 20,000 N/m. EI for Cantilever is $1 \times 10^7 \text{ Nm}^2$. EI for SSB is $2 \times 10^7 \text{ Nm}^2$.



sol- If AB is not resting on CD,

$$\Delta = \frac{wL^4}{8EI} = \frac{20 \times 2^4}{8 \times 10,000} = 4 \times 10^{-3} \text{ m.}$$

Beam AB could be pushed up by a force 'P'.

$$\therefore \Delta_1 = \frac{PL^3}{3EI} = \frac{P \times 2^3}{3EI}$$

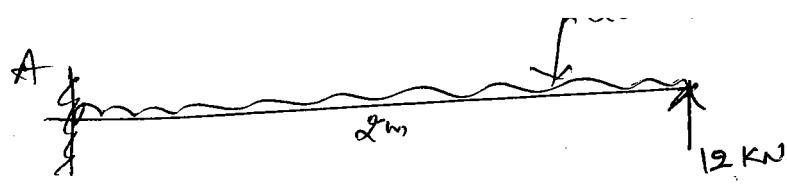
force 'P' on CD could bring down B, by Δ_2 ,

$$\Delta_2 = \frac{PL^3}{48EI} = \frac{P \times 4^3}{48 \times QEI}$$

$$\therefore 4 \times 10^{-3} = \frac{20 \times 2^4}{8EI} = \frac{P \times 2^3}{3EI} + \frac{P \times 4^3}{48 \times QEI}$$

$$\frac{20 \times 2^4}{8} = P \left(\frac{2}{3}\right) (5)$$

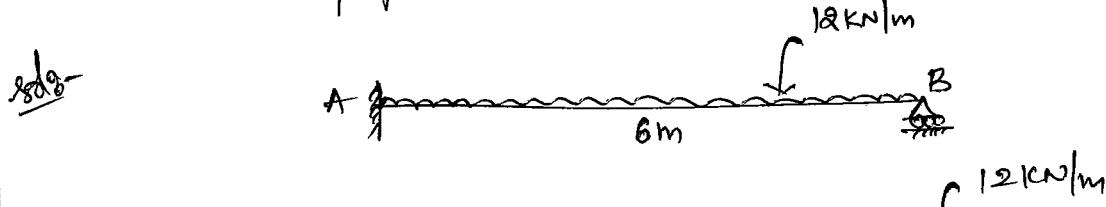
$$P = \frac{20 \times 2 \times 3}{10} = \underline{\underline{12 \text{ KN}}}$$



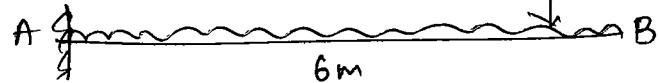
$$M_A = 12 \times 2 - 20 \times 2 \times 1 = 24 - 40 = -16 \text{ kNm}$$

$$\Delta_B = 4 \times 10^{-3} \text{ m} = 4 \text{ mm}$$

~~Supply Deflection~~
prob:- A cantilever of 6m length carries an UDL of 12 kN/m over the full span. If the free end is supported by a prop, find the reaction at the prop and also draw the S.F and B.M diagrams?



Removing prop-



$$\Delta_{B,L} = \frac{C O L^4}{8 E I} = \frac{12 \times 6^4}{8 E I}$$

Applying Redundant :-

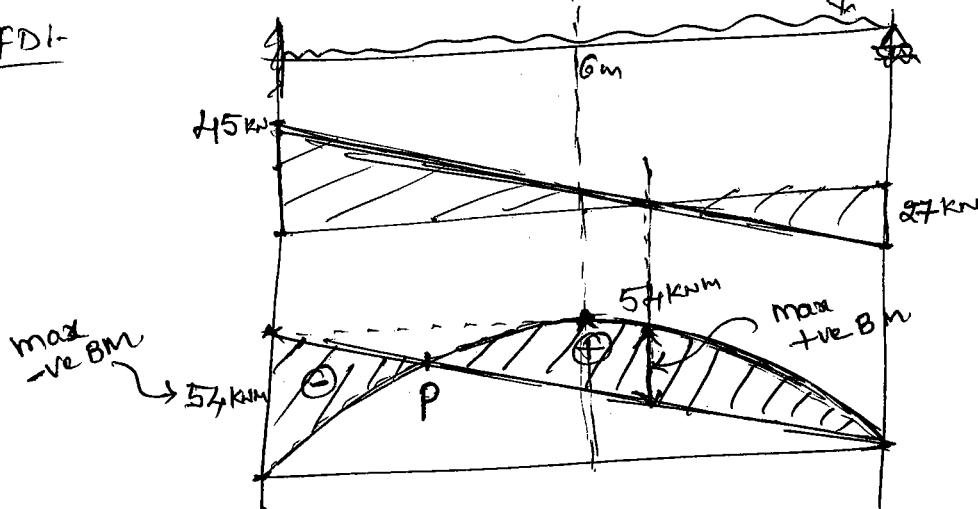
$$\Delta_{B,R} = \frac{R_B \alpha L^3}{3 E I} \quad (\text{ve})$$

from Consistency :- $\Delta_B = 0 \quad \Delta_{B,L} + \Delta_{B,R} = 0$

$$\frac{C O L^4}{8 E I} - \frac{R_B L^3}{3 E I} = 0 \Rightarrow \frac{12 \times 6^4}{8 E I} = \frac{R_B \times 6^3}{3 E I}$$

$$R_B = 27 \text{ kN} \quad R_A = 45 \text{ kN} ; M_A = -54 \text{ kNm}$$

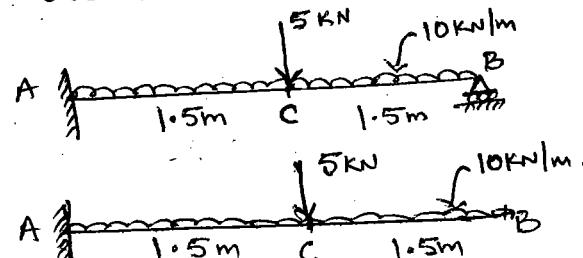
SFD &



$\therefore P$ - point of contra flexure

May/Jun 2015 (a) prob:- Analyse the propped cantilever beam shown below. Draw S.F.D. and B.M.D. Assume EI constant throughout? (8M)

(c)
-1



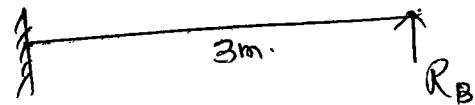
Ans:- Removing prop :-

$$\Delta_{B,L} = \frac{\omega L^4}{8EI} + \frac{\omega L^3}{3EI} + \Theta_c \times (L_2)$$

$$= \frac{10 \times 3^4}{8EI} + \frac{5 \times (1.5)^3}{3EI} + \frac{5 \times (1.5)^2}{2EI} \times (1.5)$$

$$= \frac{10 \times 25}{EI} + \frac{5.625}{EI} + \frac{8.4375}{EI} = \frac{115.3125}{EI}$$

Applying Redundant,



$$\Delta_{BR} = \frac{R_B \times 3^3}{3EI} = \frac{9R_B}{EI} \text{ (re)}$$

$$\text{from consistency, } \Delta_B = 0 \Rightarrow \Delta_{BL} + \Delta_{BR} = 0$$

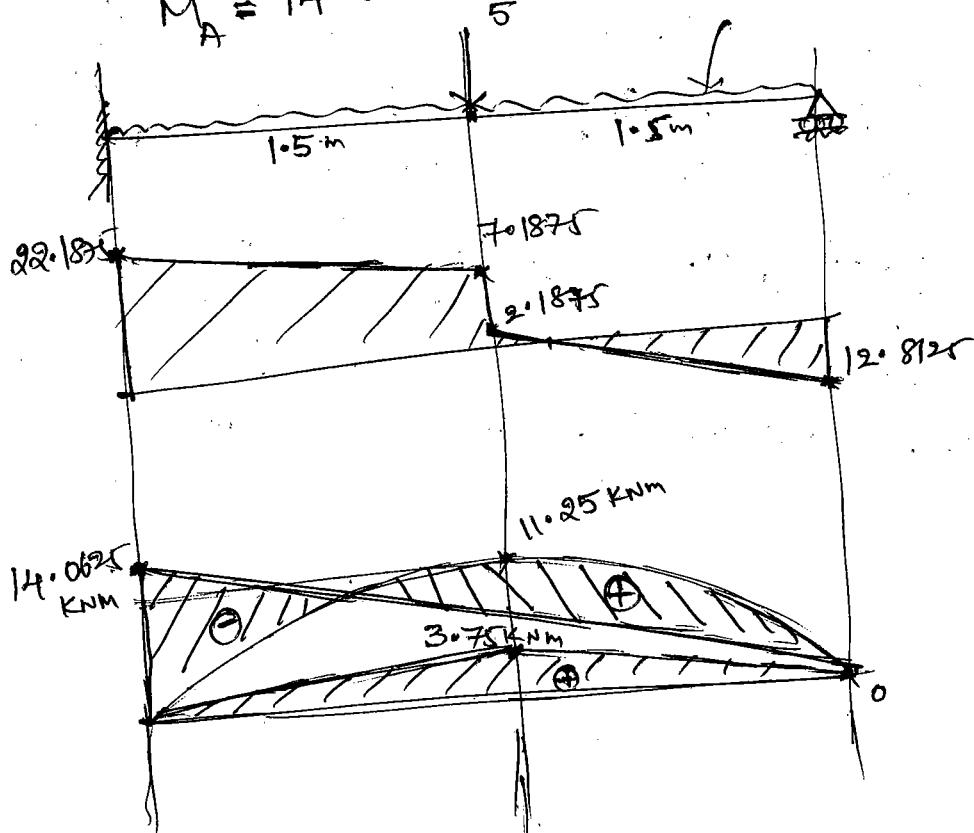
$$\therefore \frac{115.3125}{EI} = \frac{R_B \times 9}{EI} \Rightarrow \boxed{R_B = 12.8125 \text{ KN}}$$

$$\boxed{R_A = 82.1875 \text{ KN}}$$

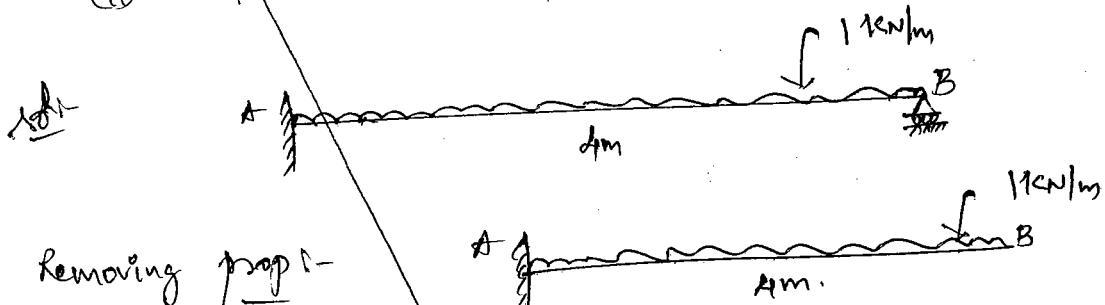
$$\sum \Delta M_A = 0 \quad M_A + 12.8125 \times 3 - 5 \times 1.5 - 10 \times 3 \times 1.5 = 0$$

$$M_A = 14.0625 \text{ KNM}$$

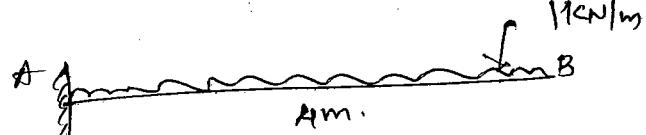
SFDI-



(b) A cantilever of length 4m carries a UDL of 1 kN/m length over the entire length. Free end of the cantilever is supported on a prop. If $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 10^8 \text{ mm}^4$, then (i) find the prop reaction (8M)
 (ii) deflection at the centre of cantilever? (8M)

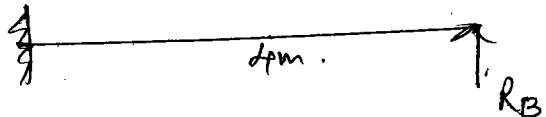


Removing prop -



$$\Delta_{B,L} = \frac{col^4}{8EI} = \frac{1 \times 4^4}{8EI}$$

Applying Redundant,



$$\Delta_{B,R} = \frac{R_B^4}{3EI} \text{ (new)}$$

from consistency

$$\Delta_R = 0 = \Delta_{B,L} + \Delta_{B,R}$$

$$\therefore \frac{4^4}{8EI} = \frac{R_B^4 / 3}{3EI} \Rightarrow R_B = \frac{3}{2} = 1.5 \text{ kN}$$

$$\Delta_c = \Delta_{c,UDL} + \Delta_{c,PL}$$

$$= \frac{col^4}{8EI} + \frac{R_B \times 1^3}{3EI} + \frac{R_B^2 \times (4/2) \times (4/2)}{8EI}$$

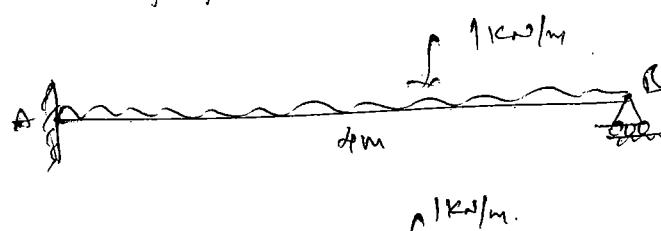
$$= \frac{1 \times (2)^4}{8EI} + \frac{(1.5) \times 2^3}{3EI} + \frac{1.5 \times (2)^2 \times (2)}{8EI} = 0.6 \text{ mm (downward)}$$

$$= \frac{1}{EI} [2 + 4 + 6] = \frac{18}{EI} = \frac{18 \times 10^3 \times (10^3)^3}{2 \times 10^5 \times 10} = 0.6 \text{ mm (downward)}$$

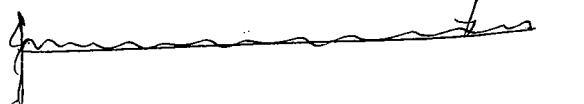
[Mansell Reciprocal Theorem]

$\delta_C = 0$

(b) A cantilever of length 4m carries a UDL of 1 kN/m length over the entire length. The free end of the cantilever is supported on a prop. If $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 10^8 \text{ mm}^4$, then find (i) the prop reaction (ii) deflection at centre? 5(i)-I

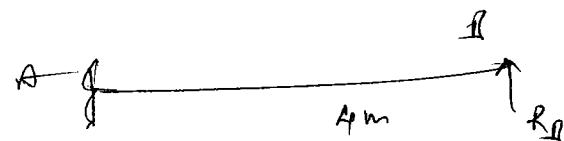


removing prop:-



$$\Delta_{BL} = \frac{COL^4}{8EI} = \frac{1 \times 4^4}{8EI} (\text{+ve})$$

Applying Redum Law

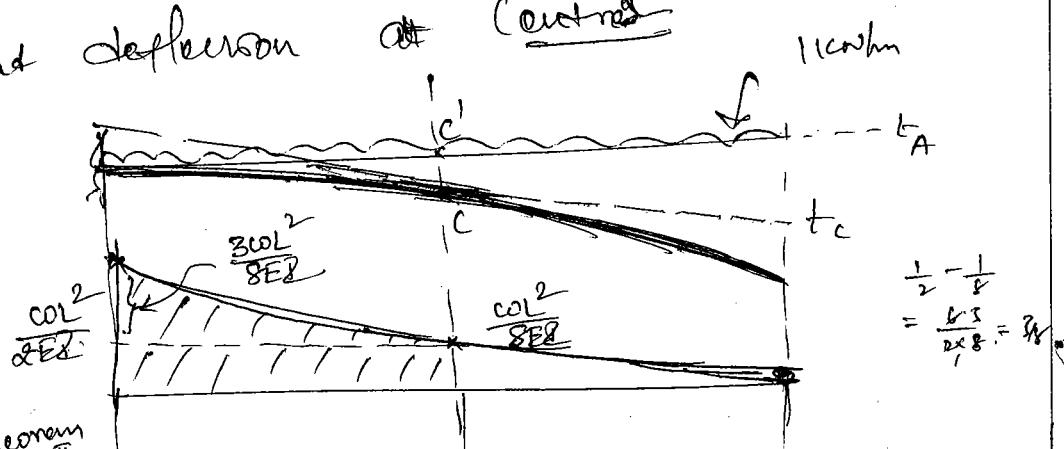


$$\Delta_{BR} = \frac{R_B L^3}{3EI} = \frac{R_B (2L)^3}{3EI} (-\text{ve})$$

from consistency cond'n Δ_B^{SO}

$$\frac{1 \times 4^4}{8EI} = \frac{R_B \times 4^3}{3EI} \Rightarrow R_B = 1.5 \text{ kN}$$

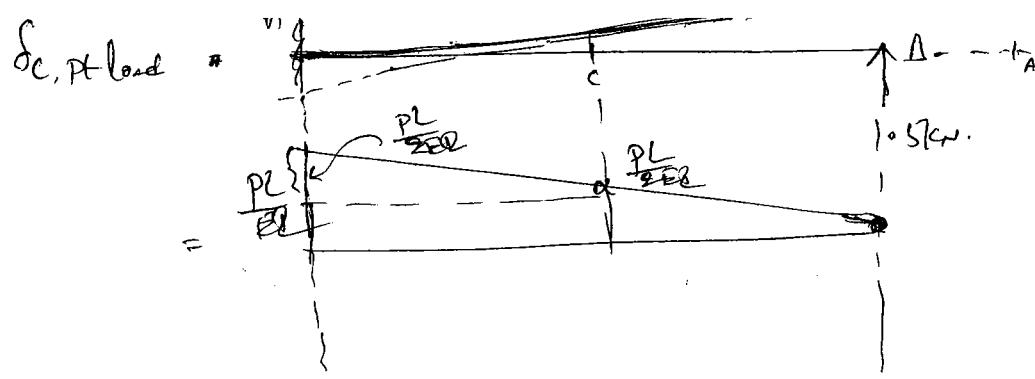
To find deflection at Centre



Mother's theorem - II

$$\therefore \delta_{C, UD} = \left(\frac{COL^2}{8EI} \right) \times \frac{L}{2} \times \frac{L}{4} + \left(\frac{1}{2} \times \frac{L}{2} \times \frac{3COL^2}{8EI} \right) \times \left(\frac{3}{4} \times \frac{L}{2} \right)$$

$$= \left(\frac{COL^4}{16EI} + \frac{3COL^4}{16 \times 8EI} \right) = \frac{COL^4}{64EI} \left(1 + \frac{3}{2} \right) = \frac{COL^4}{64EI} \times \frac{5}{8} = \frac{5COL^4}{128EI}$$

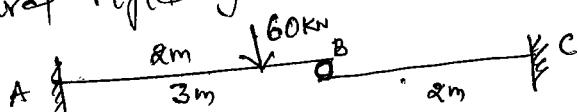


$$\begin{aligned}\delta_{C, \text{pt load}} &= \frac{PL^3}{48EI} \times \frac{L}{2} \times \frac{L}{4} + \left(\frac{1}{2} \times \frac{L}{2} \times \frac{PL}{2EI} \right) \times \left(\frac{2}{3} \times \frac{L}{2} \right) \\ &= \frac{PL^3}{16EI} + \frac{PL^3}{48EI} = \frac{PL^3}{48EI} \left(\frac{1}{4} + \frac{1}{6} \right) \\ &= \frac{PL^3}{48EI} \times \frac{105}{24} = \underline{\underline{\frac{5PL^3}{48EI}}}\end{aligned}$$

$\therefore \delta_C = \delta_{C, \text{UDL}} - \delta_{C, \text{pr.}}$

$$= 0.5 - 0.5 = \underline{\underline{0 \text{ mm}}}$$

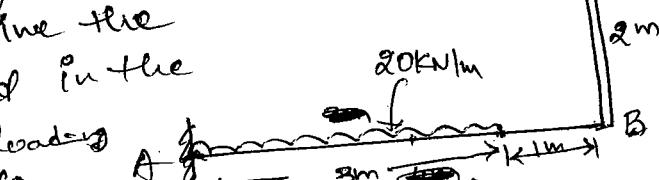
- SSBI
- ① Beam AB of span 3m rest over another beam BC of span 2m as shown below. Find the reactions at the supports A and C, given that the flexural rigidity of beam AB is twice that of BC?



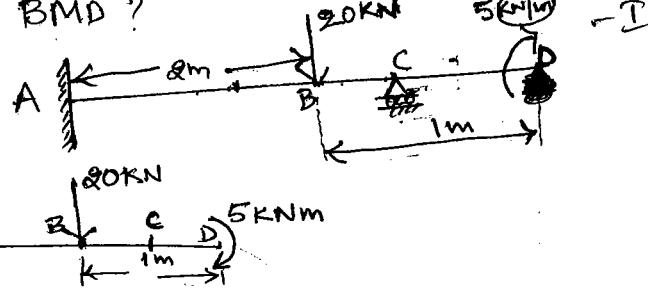
$$\begin{aligned}\text{Ans} \quad V_A &= 40.046 \text{ kN} \\ M_A &= 61.38 \text{ kNm} \\ V_C &= 19.54 \text{ kN} \\ M_C &= 39.08 \text{ kNm}\end{aligned}$$

- ② The cantilever beam shown below, of span 4m is supported by a 2m long, 3mm diameter wire. Determine the force developed in the wire due to loads shown in the figure. If $EI = 5000 \text{ KN/m}^3$ and $E_{\text{wire}} = 200 \text{ KN/mm}^2$.

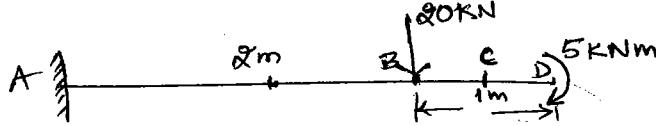
Ans- $P = \underline{\underline{10.3 \text{ kN}}}$



Set-2 (a) Determine the reactions of the propped cantilever beam and draw SFD and BMD?



Removing prop-



$$\begin{aligned}\Delta_{c,L} &= \frac{20 \times (2)^3}{3EI} + \frac{20 \times 2^2}{2EI} \times (0.5) + () \\ &= [\Delta_B + \theta_B \times BC] + \frac{5 \times 2.5^2}{2EI} + \left(\frac{5 \times 2.5}{EI} \right) \times 0.5 \\ &= " + [\Delta_C + \theta_C \times CD] \text{ moment.} \\ &= \frac{160}{3EI} + \frac{20}{EI} + \frac{15.625}{EI} + \frac{6.25}{EI} \\ &= \frac{95.2083}{EI}\end{aligned}$$

$$\Delta_{c,R} = \frac{R_c \times (2.5)}{3EI} = \frac{R_c \times 5.2083}{EI} \leftarrow \text{Ans}$$

$$\text{from Consistency, } \Delta_c = 0 \Rightarrow \Delta_{c,L} + \Delta_{c,R} = 0$$

$$\frac{95.2083}{EI} = \frac{R_c \times 5.2083}{EI}$$

$$R_c = 18.28 \text{ kN}$$

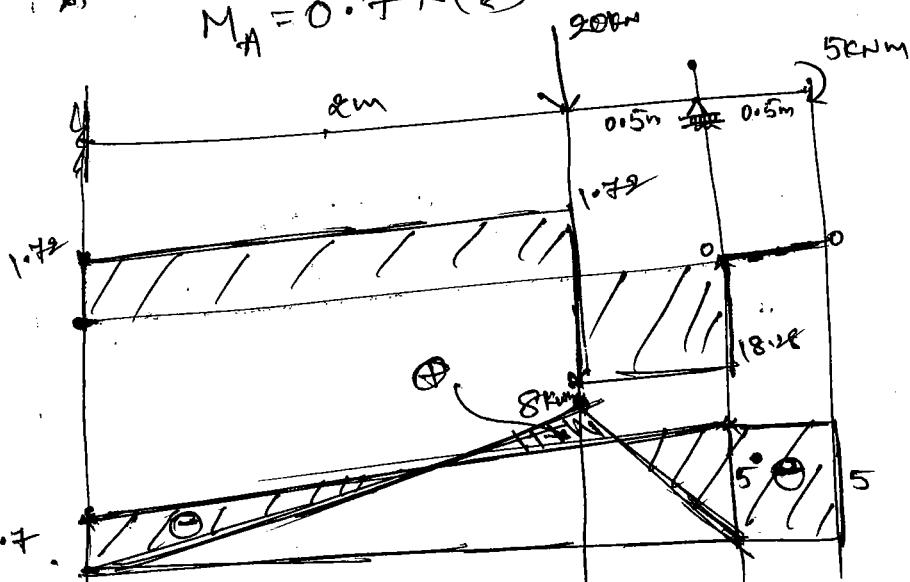
$$R_A = 20 - 18.28 = 1.72 \text{ kN}$$

$$R_A = 1.72 \text{ kN}$$

$$\sum M_A = 0 \Rightarrow M_A + 18.28 \times 2.5 = 20 \times 2 - 5 = 0$$

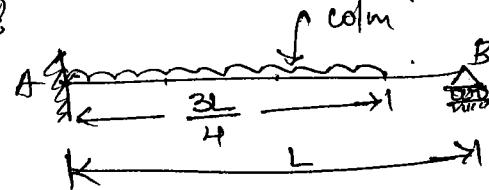
$$M_A = 0.7 \text{ kNm}$$

SFD

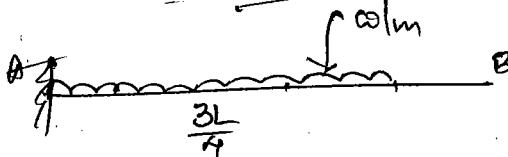


BMD

Set prop A propped cantilever beam of length L is subjected to UDL of w/cm length over three-fourth of its span from the fixed support. Determine the prop reaction and sketch the BMD?



Sol Remove the prop +



$$\Delta_{B,L} = \frac{w(3L/4)^4}{8EI} +$$

$$\frac{w(3L/4)^3}{6EI} + (L/4) \times \frac{3}{3}$$

$$= \frac{w(3L)^4}{EI} \left(\frac{1}{8} + \frac{1}{18} \right)$$

Applying Redundancy $\Delta_{B,R} = \frac{R_B L^3}{3EI}$ (neg)

From Consistency, $\Delta_B = 0 \quad \Delta_{B,L} + \Delta_{B,R} = 0$

$$\therefore \frac{w(3L)^4}{EI} \left(\frac{1}{8} + \frac{1}{18} \right) + \frac{R_B L^3}{3EI} = 0$$

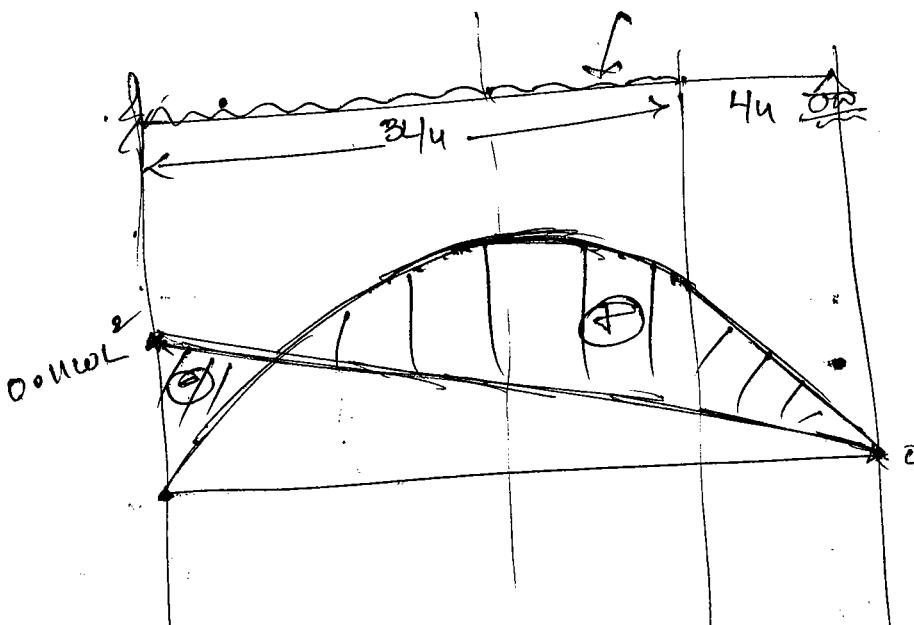
$$\frac{3w \times 3^4 L^3}{4^4} \times \left(\frac{13}{8 \times 18} \right) = R_B = \frac{351wL}{2048}$$

Sum $M_A + \frac{351wL}{2048} \times L - w \times \left(\frac{3L}{4} \right) \times \left(\frac{3L}{8} \right) = 0$

$$M_A = wL^2 (0.2815) - wL^2 (0.1714)$$

$$= wL^2 (0.1) \text{ colm}$$

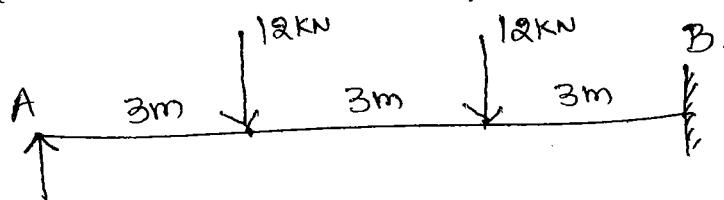
BMD



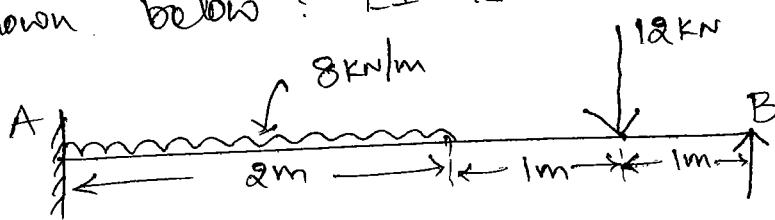
P10FF - Universt

Exercise Questions from T.S. ToM

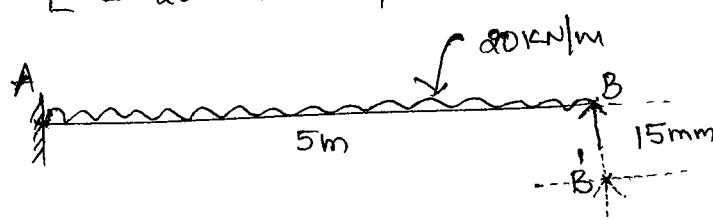
- ① What is a propped cantilever?
- ② Is it statically determinate or Indeterminate?
- ③ If it is Indeterminate, what is the degree of indeterminacy?
- ④ How is the prop reaction determined?
Explain!
- ⑤ Explain the consistent deformation method of analysing a propped cantilever.
- ⑥ Determine the prop reaction in the beam shown below? EI is constant.



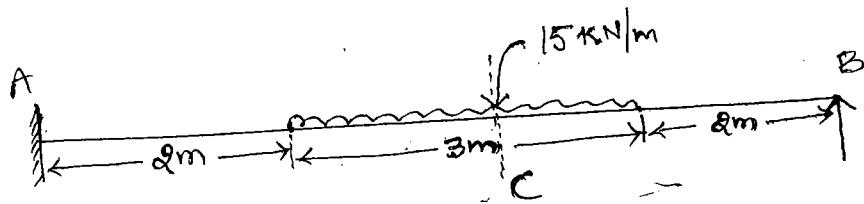
- ⑦ Using consistent deformation method, evaluate the prop reaction in the beam shown below? EI is constant.



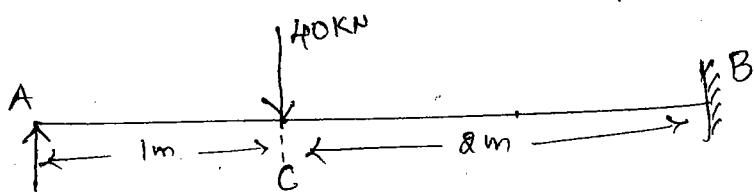
- ⑧ In the beam shown below, the prop has sunk by 15mm. calculate the prop reaction. Take $E = 200 \times 10^6 \text{ KN/m}^2$ and $I = 5 \times 10^{-6} \text{ m}^4$.



- ⑨ Find the deflection at C in the beam shown below. Take $EI = 9000 \text{ KN m}^2$.

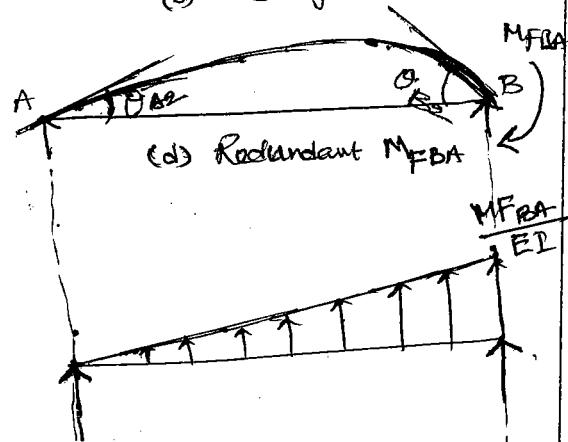
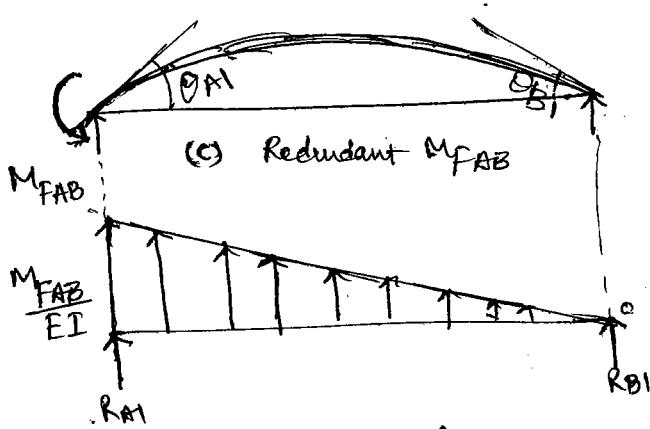
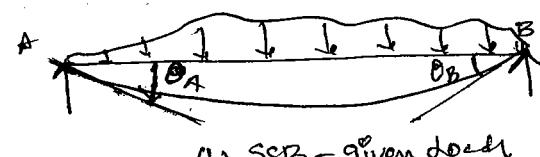
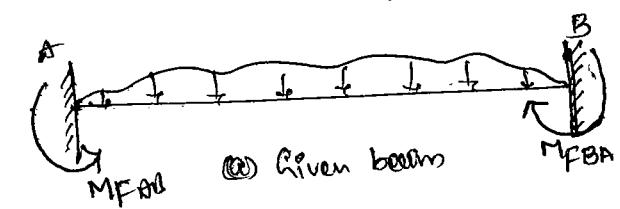


- ⑩ Determine the deflection at C in a propped cantilever shown below?



Fixed Beams

Consider the fixed beam shown below. Let the end moments developed be M_{FAB} and M_{FBA} . To find these end moments, the consistent deformation method may be used. Taking a Simply Supported beam as a basic determinate structure, the redundant forces are M_{FAB} and M_{FBA} .



from consistency:- (i) slope at A should be zero.
 (ii) " " B "

$$\theta_A - \theta_{A1} - \theta_{A2} = 0$$

$$\theta_B - \theta_{B1} - \theta_{B2} = 0$$

$$\boxed{\theta_A = \theta_{A1} + \theta_{A2}}$$

$$\boxed{\theta_B = \theta_{B1} + \theta_{B2}}$$

$$\begin{aligned}\theta_{A1} &= \left(\frac{1}{2} \times L \times \frac{M_{FAB}}{EI} \right) \times \frac{1}{3} \times \frac{1}{4} \\ &= \frac{M_{FAB} L}{3EI}\end{aligned}$$

$$\theta_{B1} = \frac{M_{FAB} L}{6EI}$$

$$\therefore \theta_A = \frac{L}{6EI} (2M_{FAB} + M_{FBA}) ; \quad \theta_B = \frac{L}{6EI} (2M_{FBA} + M_{FAB})$$

$$\begin{aligned}\theta_{A2} &= \left(\frac{1}{2} \times L \times \frac{M_{FBA}}{EI} \right) \times \frac{1}{3} \times \frac{1}{4} \\ &= \frac{M_{FBA} L}{3EI}\end{aligned}$$

$$\theta_{B2} = \frac{M_{FBA} L}{6EI}$$

Procedure :-

Step 1:- Remove the Redundants (α No's) and prepare a basic determinate beam (Simply Supported Beam) and calculate slope at both the ends θ_A, θ_B due to given loads.

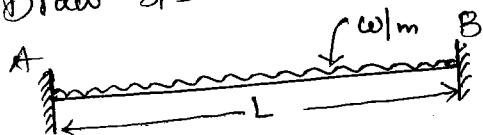
Step 2:- Apply the Redundants and find the slopes at the ends of the SSB.

Step 3:- Apply consistency conditions to solve the redundants (M_{FAB}, M_{FBA}).

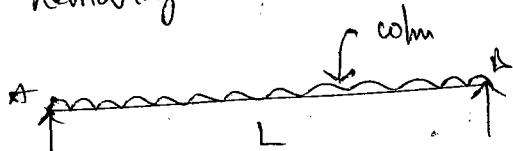
Step 4:- Apply equilibrium equations to solve for the support reactions.

Step 5:- Draw SFD and BMD and also the required deflections.

prob:- Find the fixed end moments developed in the fixed beam shown below? Draw SFD and BMD?



sol:- Removing Redundants,

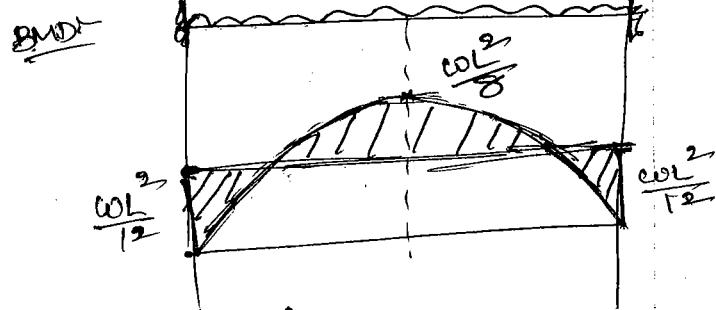
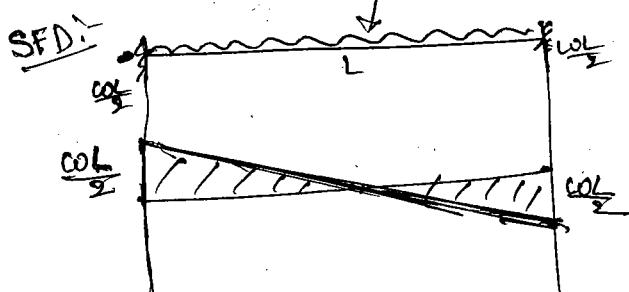


$$\theta_A = \theta_B = \frac{c o l^3}{24 EI}$$

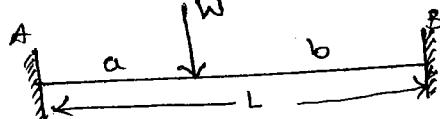
from Consistency, $\theta_A = \frac{1 \times L}{6EI} (\alpha M_{FAB} + M_{Pext})$

$$\frac{c o l^3}{24 EI} = \frac{L}{6EI} (\beta M_{FAB}) \quad (\because \text{due to Symmetry} \\ M_{FAB} = M_{FBA})$$

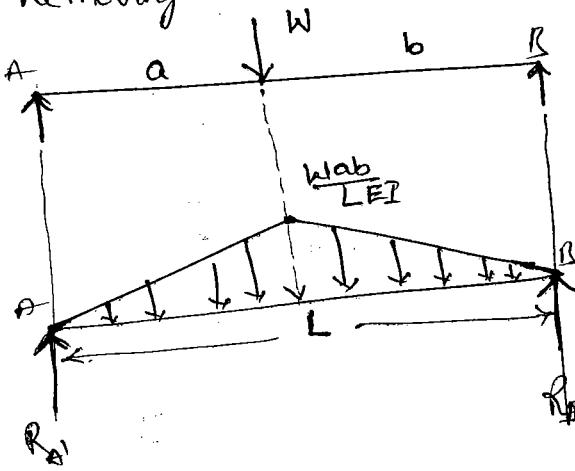
$\therefore M_{FAB} = M_{FBA} = \frac{c o l^2}{12} \text{ "}$



prob. A linear beam of span L is subjected to a concentrated load of W at a distance 'a' from end A. Determine end moments? (Q-II)



sdr Removing the Redundants.



$$\Theta_A = R_A' = \left(\frac{1}{2} \times \frac{1}{L} \times \frac{Wab}{LEI} \right) \times$$

$$\left(\frac{L+b}{3} \right) \times \frac{1}{L}$$

$$= \frac{Wab}{6EI} \left(\frac{L+b}{L} \right)$$

$$\text{likewise } \Theta_B = \frac{Wab}{6EI} \left(\frac{L+a}{L} \right)$$

Applying Redundants,

$$\Theta_A = \frac{L}{6EI} (\Delta M_{FAB} + M_{FBA})$$

$$\Theta_B = \frac{L}{6EI} (\Delta M_{FBA} + M_{FAB})$$

$$\text{from Consistency: } \frac{Wab}{6EI} \left(\frac{L+b}{L} \right) = \frac{L}{6EI} (\Delta M_{FAB} + M_{FBA})$$

$$\begin{aligned} L \times \left(\frac{Wab}{L^2} \frac{(L+b)}{L} \right) &= \Delta M_{FAB} + M_{FBA} \times 2 \\ \cancel{L} \times \frac{Wab}{\cancel{L^2}} \frac{(L+a)}{L^2} &= \Delta M_{FAB} + \cancel{\Delta M_{FBA}}. \end{aligned}$$

$$\frac{Wab}{L^2} (2a + 2b - L - a) = 3M_{FAB}$$

$$\frac{Wab}{L^2} (a + b + 2b - a) = 3M_{FAB}$$

$$\frac{Wab}{L^2} (3b) = 3M_{FAB}$$

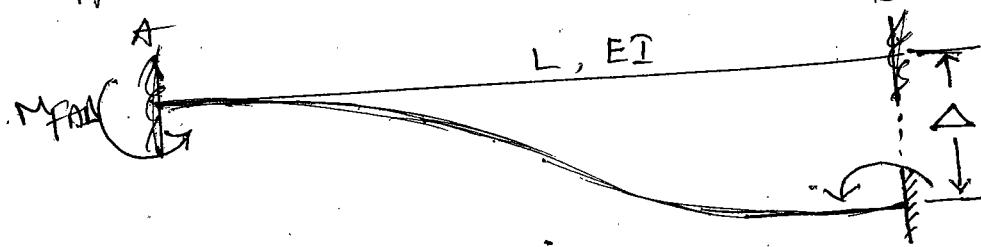
$$\therefore M_{FAB} = \frac{Wab^2}{L^2}$$

$$M_{FBA} = \frac{Wba^2}{L^2}$$

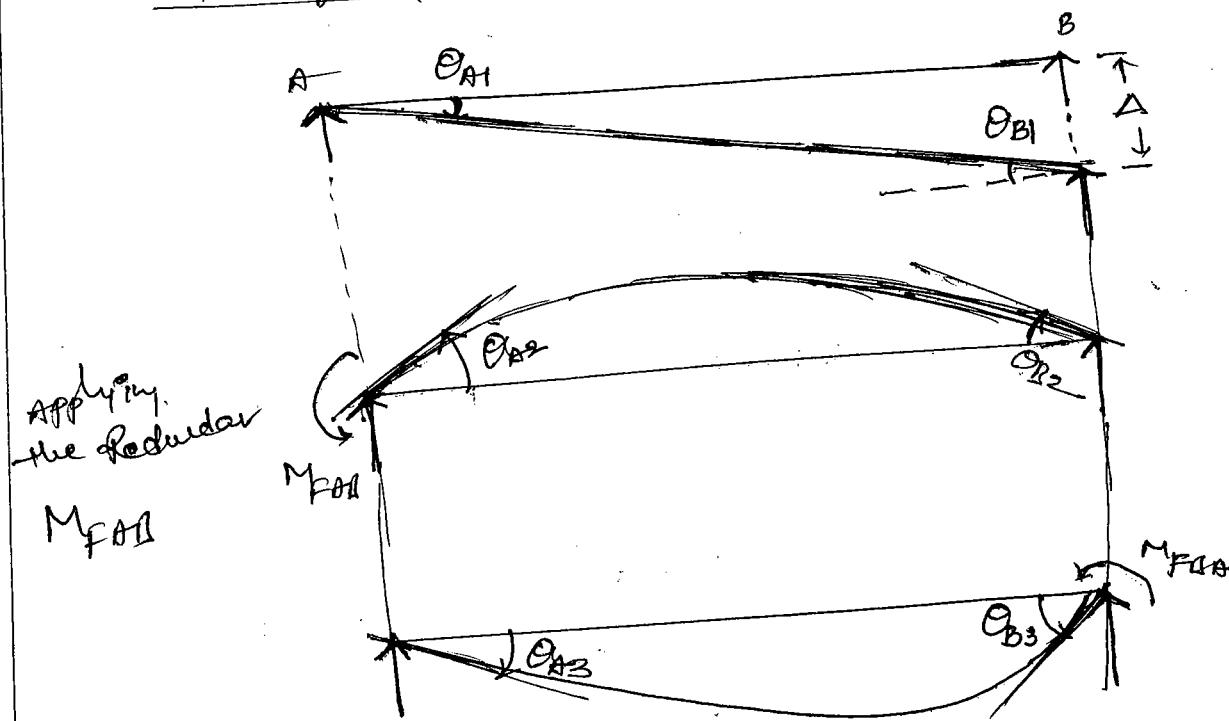
\therefore if $a = b = L/2$

$$\begin{aligned} M_{FAB} &= M_{FBA} \\ &= \frac{WL}{8} \end{aligned}$$

In a fixed beam of span L and flexural Rigidity EI when the right hand side support settles down by Δ ?



Step 1 Removing the Redundants



$$\theta_{A1} = \theta_{B1} = \frac{\Delta}{L} \quad (\text{clockwise})$$

$$\theta_{A2} = \frac{M_{FAB} L}{3EI} \quad (\text{cw}) ; \quad \theta_{B2} = \frac{M_{FAB} L}{6EI}$$

$$\theta_{A3} = \frac{M_{FBAL}}{6EI} \quad (\text{ccw}) ; \quad \theta_{B3} = \frac{M_{FBAL}}{3EI} \quad (\text{ccw})$$

from consistency, $\theta_{A2} = 0$; $\theta_{B2} = 0$

$$\frac{\Delta}{L} - \frac{M_{FAB} L}{3EI} + \frac{M_{FBAL}}{6EI} = 0 \quad \text{--- (1)}$$

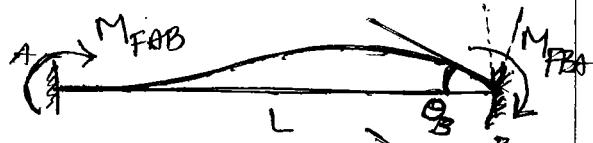
$$\frac{\Delta}{L} + \frac{M_{FAB} L}{6EI} - \frac{M_{FBAL}}{3EI} = 0 \quad \text{--- (2)}$$

from (1) & (2) $M_{FAB} = M_{FBA} \Rightarrow \text{Substituting in (1) or (2)}$

$$M_{FAB} = M_{FBA} = \frac{6EI\Delta}{L^2} //$$

(3)-II

Prob. - Determine the fixed end moments developed in a fixed beam of length L and flexural Rigidity EI when the right hand side support is rotated by an angle θ ?



Sol: Removing the redundants applying M_{FAB} ,

$$\theta_A = \frac{M_{FBA}L}{6EI}$$



$$\theta_B = \frac{M_{FBA}L}{3EI}$$



$$\theta_{A2} = \frac{M_{FAB}L}{3EI} ; \theta_{B2} = \frac{M_{FAB}L}{6EI}$$

\therefore from consistency, $\theta_A = 0$ $\theta_A + \theta_{A2} = 0$

$$-\frac{M_{FBA}L}{6EI} \neq \frac{M_{FAB}L}{3EI} = 0 \Rightarrow \frac{M_{FBA}L}{6EI} = \frac{M_{FAB}L}{3EI}$$

$$M_{FAB} = \frac{M_{FBA}}{2}$$

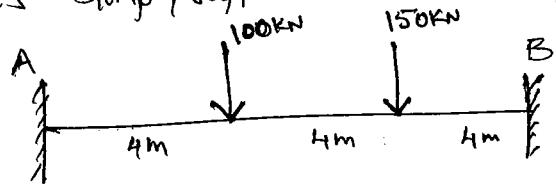
$$\theta_B = \theta_B - \theta_{B2} = \frac{M_{FBA}L}{3EI} - \frac{M_{FAB}L}{6EI}$$

$$\begin{aligned} \theta_B &= \frac{M_{FBA}L}{3EI} - \frac{M_{FBA}L}{2 \times 6EI} \\ &= \frac{M_{FBA}L}{3EI} \left(\frac{1}{3} - \frac{1}{12} \right) = \frac{M_{FBA}L}{3EI} \times \frac{1}{4} \end{aligned}$$

$$\therefore M_{FBA} = \frac{4EI\theta_B}{L}$$

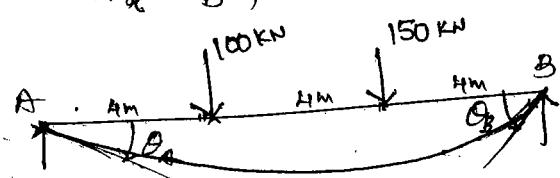
$$\therefore M_{FAB} = \frac{2EI\theta_B}{L}$$

loads at a distance of 4m and 8m from left end respectively. Find the fixed end moments under the loads when the beam is Simply Supported. Draw BMD.



~~(S1)~~ Removing Redundants

M_A & M_B ;

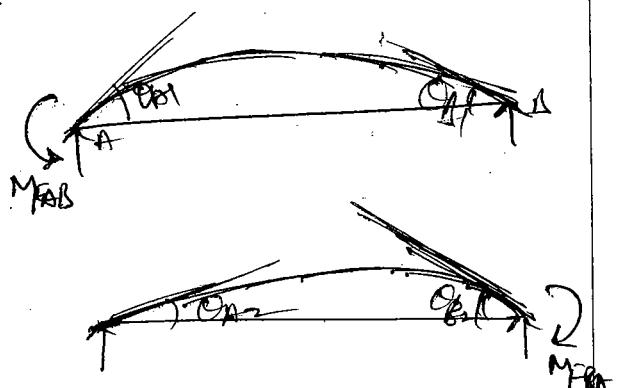


$$\theta_A = \frac{W_{ab}(L+b)}{6EI}; \theta_B = \frac{k_{ab}(L+a)}{6EI}$$

$$\begin{aligned} \theta_A &= \frac{100 \times 4 \times 8 (12+8)}{6 \times 12 \times EI} + \frac{150 \times 8 \times 4 (12+4)}{6 \times 12 \times EI} \\ &= \frac{888.89}{EI} + \frac{1066.67}{EI} = \frac{1955.56}{EI} \end{aligned}$$

$$\begin{aligned} \theta_B &= \frac{100 \times 4 \times 8 \times (12+4)}{6 \times 12 \times EI} + \frac{150 \times 8 \times 4 (12+8)}{6 \times 12 \times EI} \\ &= \frac{147200}{6 \times 12 \times EI} = \frac{2444.44}{EI} \end{aligned}$$

Applying Redundants



From consistency,

$$\theta_A = \frac{L}{6EI} (\alpha M_{FAB} + M_{FBa})$$

$$\theta_B = \frac{L}{6EI} (M_{FAB} + \alpha M_{FBA})$$

$$\frac{1955.56}{EI} = \frac{L^2}{6EI} (\alpha M_{FAB} + M_{FBa})$$

$$\frac{2444.44}{EI} = \frac{L^2}{6EI} (\alpha M_{FAB} + \alpha M_{FBA})$$

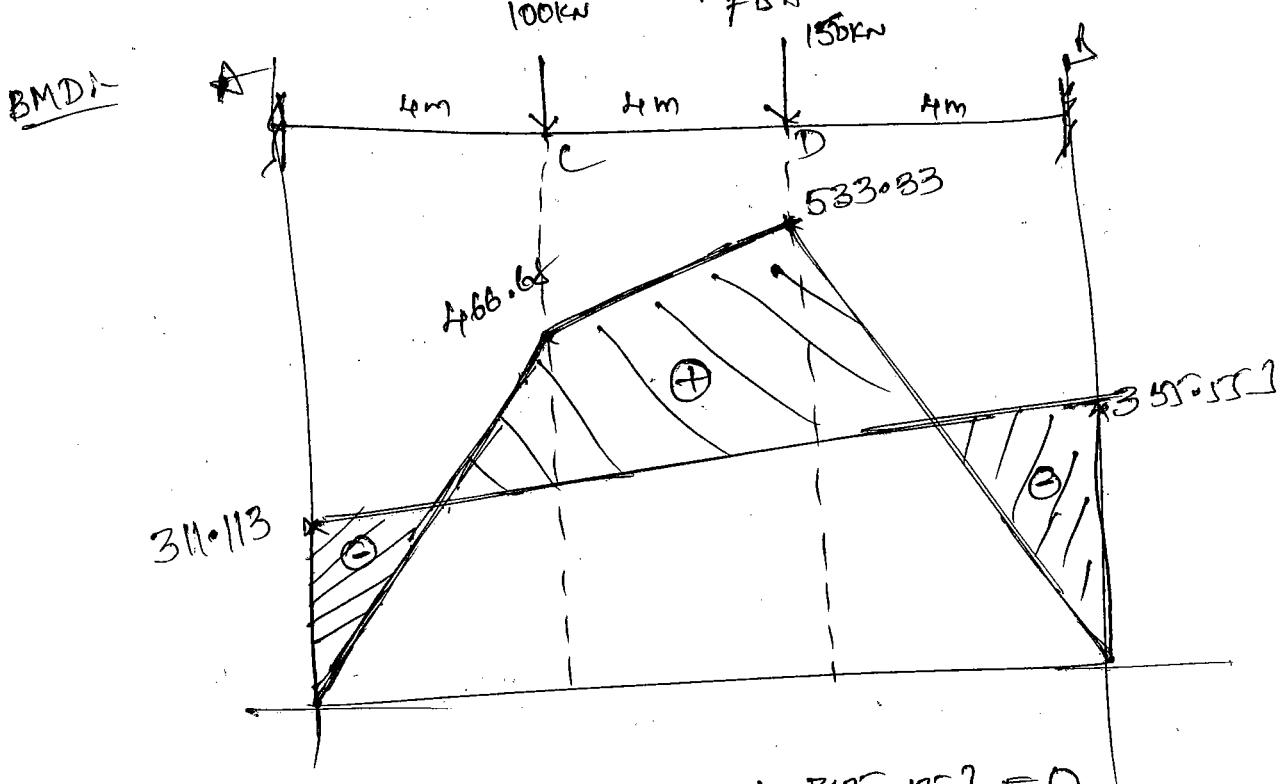
$$(1955.56 = 4M_{FAB} + 2M_{FBA}) \times 2$$

$$2044.44 = 2M_{FAB} + 4M_{FBA}$$

$$3911.12 = 8M_{FAB} + 4M_{FBA}$$

$$1866.68 = 6M_{FAB} \Rightarrow M_{FAB} = 311.113 \text{ kNm}$$

$$M_{FBA} = 355.553 \text{ kNm}$$



$$R_A \times 12 - 100 \times 8 - [50 \times 4 - 311.113 + 355.55] = 0$$

$$R_A = 112.963 \text{ kN} \quad \left. \right\} \text{including FBM}$$

$$R_B = 137.037 \text{ kN}$$

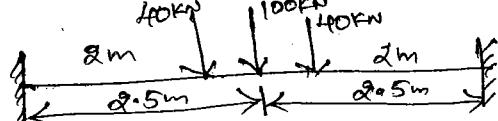
$$R_A = \frac{100 \times 8}{12} + \frac{150 \times 4}{12} = 116.67 \text{ kN}$$

$$R_B = (+) + (-) = 133.33 \text{ kN}$$

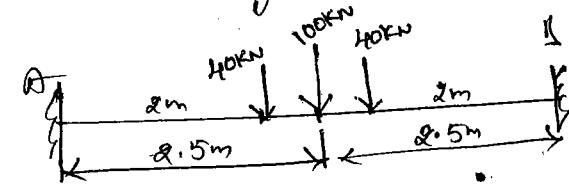
$$\text{SS, BM}_C = 116.67 \times 4 = 466.68 \text{ kNm}$$

$$\text{SS, BM}_D = 133.33 \times 4 = 533.33 \text{ kNm}$$

prob.: A fixed beam is shown below. Solve the beam and draw SFD and BMD?



1st Removing Redundants,



$$\Theta_A = \Theta_B = \frac{100 \times (5)}{16 EI}$$

$$+ \frac{40 \times 2 \times 3 \times (5+3)}{64 EI} + \frac{40 \times 2 \times 3 \times (5+3)}{64 EI}$$

$$= \frac{1}{EI} \left[\frac{156.25}{781.25} + 64 + 56 \right] = \frac{276.25}{EI} = \frac{276.25}{EI}$$

Applying Redundants, $\Theta_{ABA} = \frac{L}{6EI} (M_{FAB} + M_{FBA})$

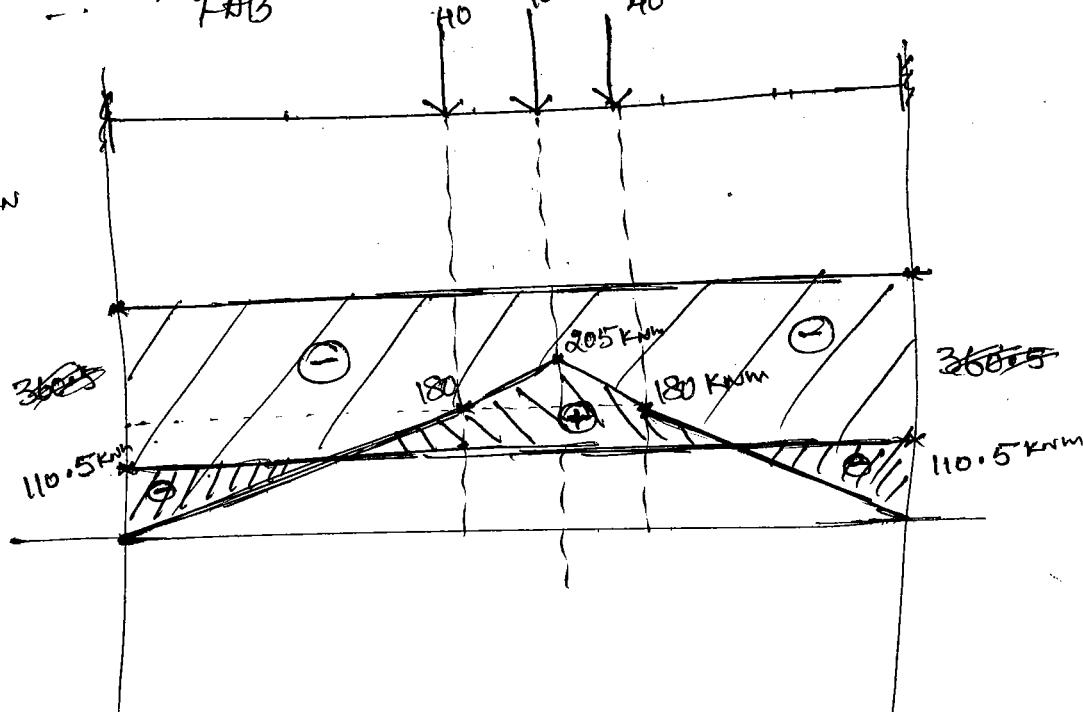
$$\Theta_{ABA} = \frac{L}{6EI} (M_{FAB} + M_{FBA})$$

from Consistency, $\frac{276.25}{EI} = \frac{5}{6EI} [2M_{FAB} + M_{FBA}]$

$$\therefore M_{FAB} = M_{FBA} \Rightarrow \frac{90.25}{EI} \times \frac{6EI}{5} = 3M_{FAB}$$

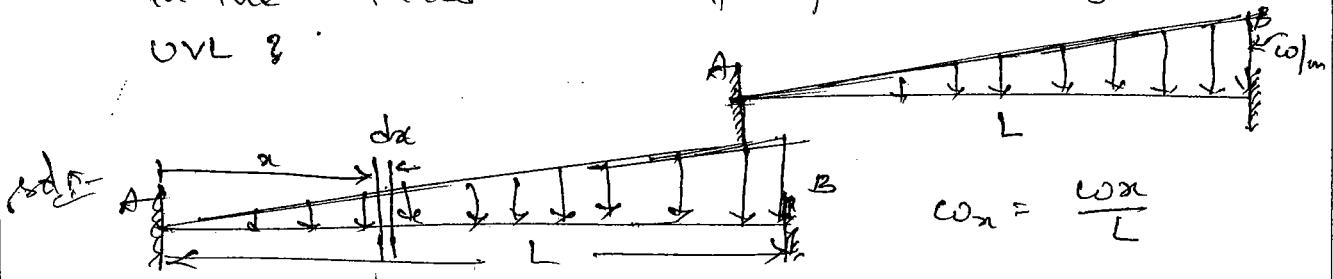
$$\therefore M_{FAB} = M_{FBA} = \frac{360.75}{5} \text{ kNm.} = 110.5$$

BMD:-
SFD
 $R_A = R_B = 90 \text{ kN}$



prob: Determine the fixed end moments developed in the fixed beam of Span L subjected to UVL?

(6)-11



$$\omega_x = \frac{\omega_0}{L}$$

$$\begin{aligned}
 M_{FAB} &= \int_0^L \frac{\omega_0}{L} \cdot x \cdot \frac{(L-x)^2}{2} dx \\
 &= \int_0^L \frac{\omega_0}{L^3} x^2 (L^2 - 2x^2 + x^4) dx \\
 &= \frac{\omega_0}{L^3} \int_0^L (x^2 L^2 + x^4 - 2Lx^3) dx \\
 &= \frac{\omega_0}{L^3} \left[L^2 \left(\frac{x^3}{3} \right) + \frac{x^5}{5} - 2L \frac{x^4}{4} \right]_0^L \\
 &= \frac{\omega_0}{L^3} \left[L^2 \left(\frac{L^3}{3} \right) + \frac{L^5}{5} - \frac{2L \cdot L^4}{4} \right] \\
 &= \frac{\omega_0}{L^3} \left[\frac{L^5}{3} + \frac{L^5}{5} - \frac{L^5}{2} \right] \\
 &= \frac{\omega_0 \cdot L^5}{L^8} \left[\frac{1}{3} + \frac{1}{5} - \frac{1}{2} \right] \\
 &= \omega L^2 \left[\frac{10+6-15}{30} \right]
 \end{aligned}$$

$$M_{FAB} = \frac{\omega L^2}{30}$$

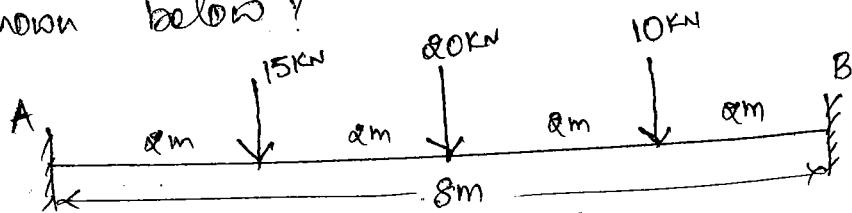
$$\begin{aligned}
 M_{FBA} &= \int_0^L \left(\frac{\omega x}{L} \right) x \frac{(L-x)}{L^2} dx \\
 &= \int_0^L \frac{\omega}{L^3} x^3 (L-x) dx \\
 &= \int_0^L \frac{\omega}{L^3} (Lx^3 - x^4) dx \\
 &= \frac{\omega}{L^3} \left[L\left(\frac{x^4}{4}\right) - \frac{x^5}{5} \right]_0^L \\
 &= \frac{\omega}{L^3} \left[\frac{L^5}{4} - \frac{L^5}{5} \right] \\
 &= \frac{\omega L^5}{L^3} \left[\frac{1}{4} - \frac{1}{5} \right] \\
 &= \omega L^2 \left[\frac{5-4}{20} \right] = \frac{\omega L^2}{20}
 \end{aligned}$$

$\boxed{M_{FBA} = \frac{\omega L^2}{20}}$

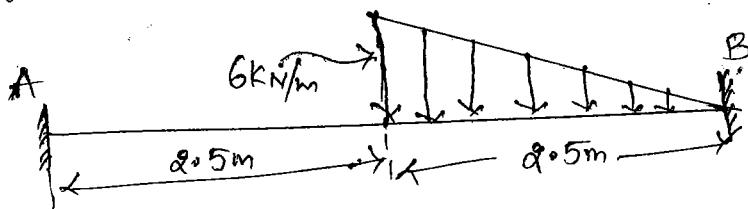
Ques. A fixed beam AB of length 3m carries a point load of 45 kN at a distance of 2m from A. If flexural rigidity of the beam is $1 \times 10^4 \text{ kNm}^2$ determine (i) Fixed end moments (ii) Deflection under the load (iii) maximum deflection

Exercise Questions from TSTM.

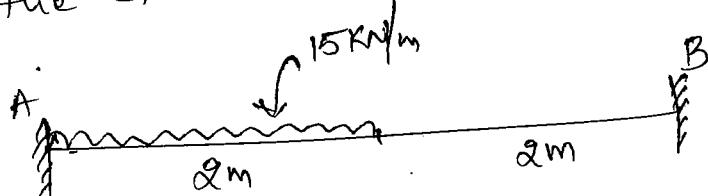
- ① what is an encastre beam?
- ② what is the degree of indeterminacy of a constrained beam?
- ③ why the ends in a fixed beam are called "Direction - fixed ends"? ($\because \theta=0; \delta=0$ fixed in direction)
- ④ Derive expressions for fixed-end moments in a fixed beam of Span 'L' carrying UDL of $w \text{ kN/m}$ by consistent deformation method?
- ⑤ Determine the fixed-end moments in the beam shown below?



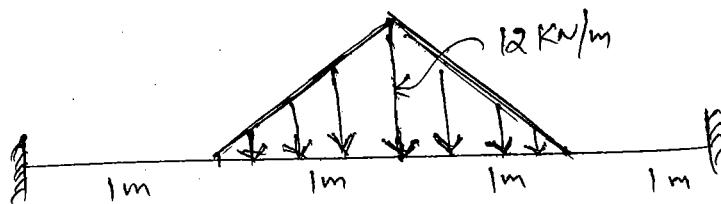
- ⑥ Evaluate the fixed end moments in the beam shown below?



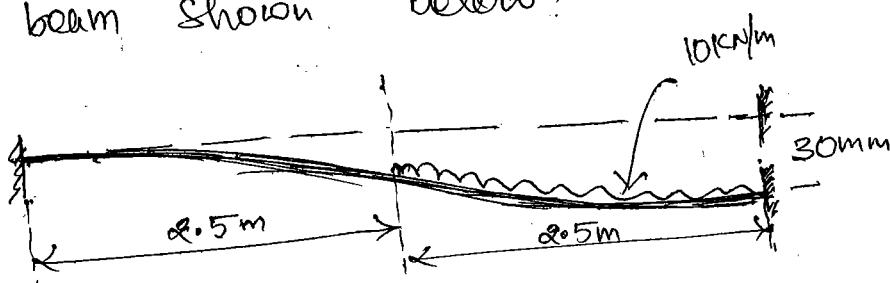
- ⑦ Draw the SFD and BMD of the beam shown below



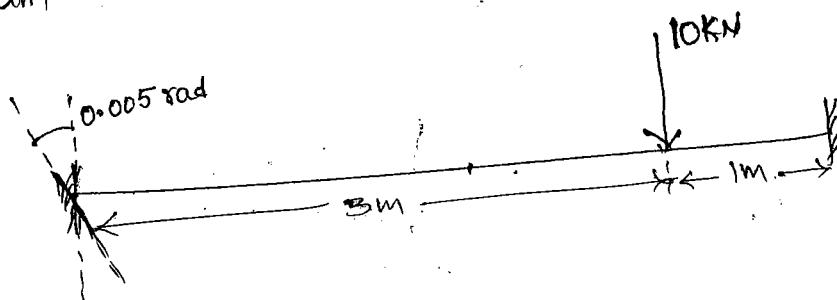
- ⑧ calculate the deflection at the centre of the beam shown below?



- ⑨ Determine the fixed end moments in the beam shown below?



- ⑩ Evaluate the fixed-end moments in the beam shown below. Take $EI = 11,000 \text{ kNm}^2$

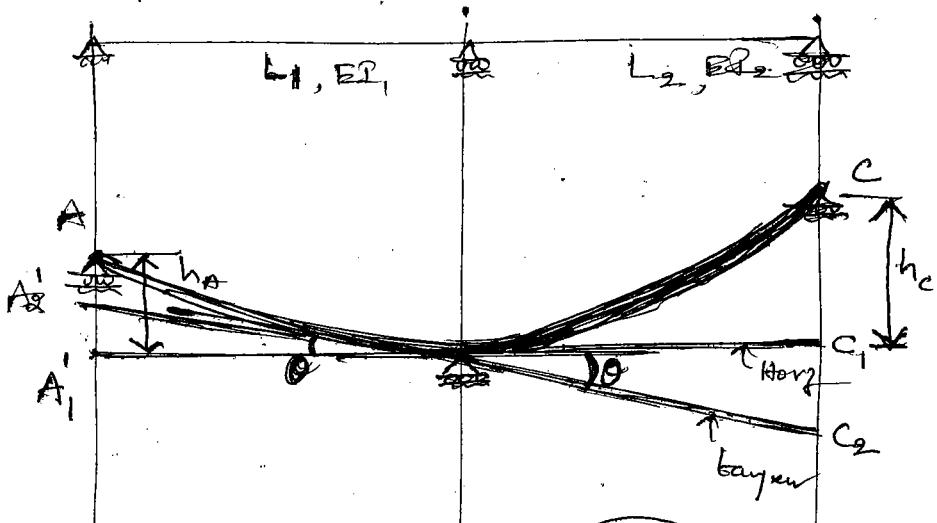


UNIT - III
Continuous Beams

CLAPEYRON'S Theorem- It is the relation between the moments at three successive supports. This is derived from the consistency condition that the slope at the middle support calculated from left span and the right span should be same.

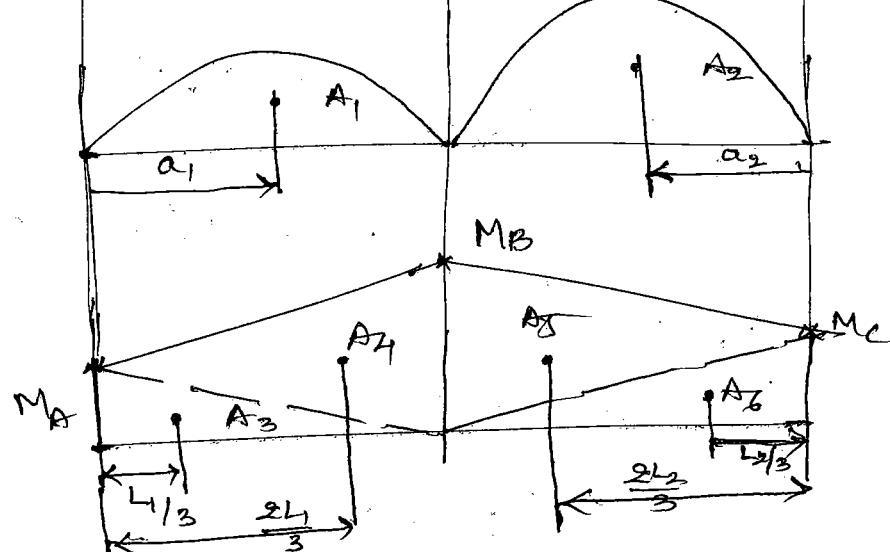
Derivation-

(a)



(b) Free BMD

(c)



$$\text{From (a)} \quad \tan \theta = \frac{A_1' A_2'}{L_1} = \frac{C_1 C_2}{L_2} \quad \text{--- (1)}$$

$$\text{But } A_1' A_2' = AA' - A_1 A_2' = h_A - A_1 A_2'$$

$= h_A - \text{Vertical intercept of } A \text{ from tangent at } B$

$= h_A - \text{moment of } (M/EI) \text{ diagram w.r.t. Bands about } A$

$$= h_A - \frac{1}{EI_1} \left[A_1 a_1 + \frac{A_3 L_1}{3} + A_4 \cdot \frac{2L_1}{3} \right]$$

$$= h_A - \frac{1}{EI_1} \left[A_1 a_1 + \left(\frac{1}{2} \times M_A \times L_1 \right) \frac{L_1}{3} + \frac{1}{2} \times M_B \times L_1 \times \frac{2L_1}{3} \right]$$

$$= h_A - \frac{1}{EI_1} \left[A_1 a_1 + \frac{M_A L_1^2}{6} + \frac{M_B L_1^2}{3} \right]$$

$$= h_A - \frac{1}{6EI_1} \left[6A_1 a_1 + M_A L_1^2 + 2M_B L_1^2 \right] \quad \textcircled{2}$$

$$\text{My: } C_{C_2} = C_{C_2} - C_C$$

= vertical intercept of \bar{C} from tangent at B
- h_c

= moment of $\left(\frac{M}{EI}\right)$ w.r.t. Bending about C - h_c

$$= \frac{1}{EI_2} \left[A_2 a_2 + A_5 \cdot \frac{2L_2}{3} + A_6 \cdot \frac{L_2}{3} \right] - h_c$$

$$= \frac{1}{EI_2} \left[A_2 a_2 + \frac{1}{2} \times M_B \times \frac{L_2}{2} \times \frac{2L_2}{3} + \frac{1}{2} \times M_C \times \frac{L_2}{3} \right] - h_c$$

$$= \frac{1}{6EI_2} \left[6A_2 a_2 + 2M_B L_2^2 + M_C L_2^2 \right] - h_c \quad \textcircled{3}$$

Substituting \textcircled{2} & \textcircled{3} in \textcircled{1}

$$\frac{h_A}{L_1} - \frac{1}{6EI_1} \left[6A_1 a_1 + M_A L_1^2 + 2M_B L_1^2 \right] =$$

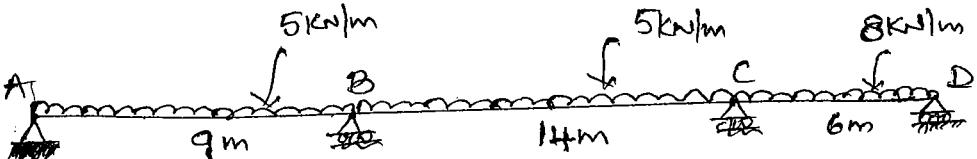
$$\frac{1}{6EI_2} \left[6A_2 a_2 + 2M_B L_2^2 + M_C L_2^2 \right] - \frac{h_c}{L_2} \times 6E$$

$$\frac{6Eh_A}{L_1} - \frac{1}{I_L} \left[6A_1 a_1 + M_A L_1^2 + 2M_B L_1^2 \right] = \frac{1}{I_L} \left[6A_2 a_2 + 2M_B L_2^2 + M_C L_2^2 \right] - \frac{6h_c}{L_2}$$

$$M_A \left(\frac{L_1}{I_L} \right) + 2M_B \left(\frac{L_1}{I_L} + \frac{L_2}{I_L} \right) + M_C \left(\frac{L_2}{I_L} \right) = -\frac{6A_1 a_1}{I_L} - \frac{6A_2 a_2}{I_L}$$

$$+ \frac{6Eh_A}{L_1} + \frac{6Eh_c}{L_2} \quad //$$

prob:- A continuous beam ABCD is simply supported over three spans. Span AB is 9m carrying an UDL of 5kN/m. Span BC is 14m carrying an UDL of 5kN/m and Span CD is 6m carrying an UDL of 8kN/m. Find the moment over supports B and C. Draw BMD?



soln Applying three moment theorem for A, B & C supports

$$M_A \left(\frac{L_1}{I_1} \right) + 2M_B \left(\frac{L_1}{I_1} + \frac{L_2}{I_2} \right) + M_C \left(\frac{1}{I_2} \right) =$$

$$-\frac{6A_1a_1}{I_1 L_1} - \frac{6A_2a_2}{I_2 L_2} + \frac{6Eh_A}{L_1} + \frac{6Eh_C}{L_2}$$

$$\because h_A = h_C = 0$$

$$\therefore EI \text{ is same throughout } I_1 = I_2 = I$$

$$\therefore M_A L_1 + 2M_B (L_1 + L_2) + M_C L_2 = -\frac{6A_1a_1}{L_1} - \frac{6A_2a_2}{L_2}$$

$$\cancel{M_A(9)} + 2M_B(9+14) + M_C(14) = -\left(\frac{6}{3} \times \frac{2}{3} \times 9 \times \frac{5 \times 9^2}{8} \times 4.5\right) \\ - \left(\frac{6}{3} \times \frac{2}{3} \times 14 \times \frac{5 \times 14^2}{8} \times 7\right)/14.$$

$$46M_B + 14M_C = \frac{-8201.25}{9} - \frac{48020}{14}$$

$$46M_B + 14M_C = -911.25 - 3430 = -4341.25 \quad \text{--- (1)}$$

Applying three moment theorem for B, C, D supports

$$M_B L_1 + 2M_C (L_1 + L_2) + M_D (L_2) = -\frac{6A_1a_1}{L_1} - \frac{6A_2a_2}{L_2}$$

$$14M_B + 2M_C (14+6) + M_D (6) = -3430 - \frac{6 \times \frac{2}{3} \times 6 \times \frac{8 \times 6^2}{8} \times 3}{14} \\ = -3430 - 432 = -3862$$

$$14M_B + 40M_C + M_D(6)$$

$$14M_B + 40M_C = -3862 \quad \text{--- (2)}$$

Solving (1) & (2)

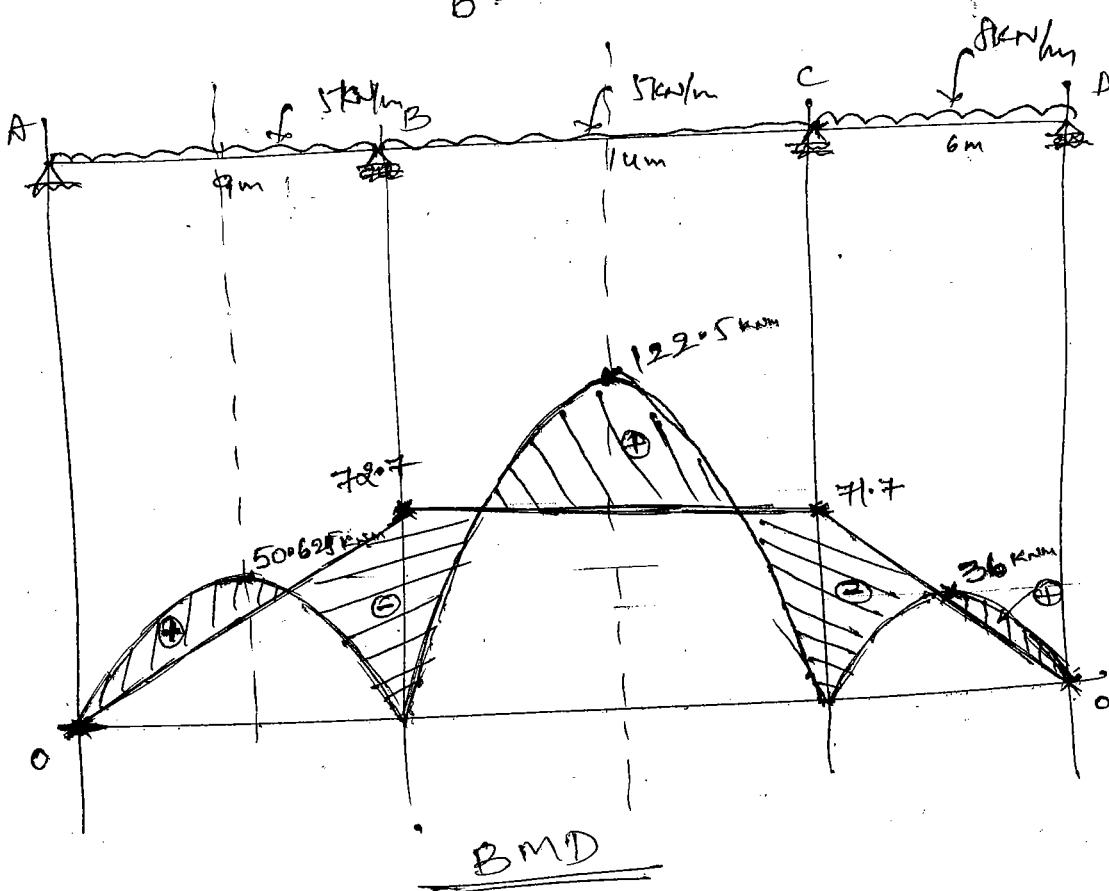
$$-M_B + 0.304 M_C = +94.575$$

$$M_B + 0.857 M_C = -275.852$$

$$2.5526 M_C = 181.482$$

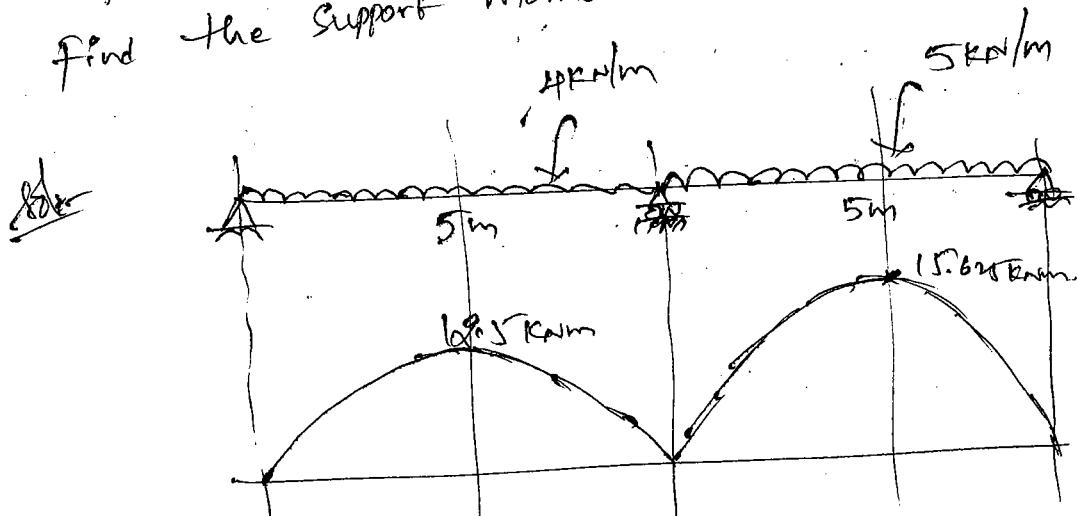
$$\therefore M_C = -71.7 \text{ kNm}$$

$$M_B = -78.7 \text{ kNm}$$



~~prob 1~~ A continuous beam ABC of length 10m is simply supported. AB and BC is of 5m length each. Span AB carries an UDL of 4kN/m and BC carries a UDL of 5kN/m. Support B sinks down by 5mm below the supports A & C. M.O. of beam is 10 mm^4 and E is 180 kN/mm^2 . Find the support moments and draw BMD?

Find the support moments and draw BMD?



Applying Clapeyron's

$$M_A \left(\frac{L_1}{I} \right) + 2M_B \left(\frac{L_1}{I_1} + \frac{L_2}{I_2} \right) + M_C \left(\frac{L_2}{I_2} \right) = -\frac{6A_1a_1}{L_1} - \frac{6A_2a_2}{L_2} + \frac{6Eh_A}{L_1} + \frac{6EI_{hc}}{L_2}$$

$\therefore EI$ is same throughout,

$$M_A \left(\frac{10}{I_1} \right) + 2M_B \left(\frac{10}{I_1} + \frac{10}{I_2} \right) + M_C \left(\frac{10}{I_2} \right) = -\frac{6A_1a_1}{L_1} - \frac{6A_2a_2}{L_2} + \frac{6EI_{hc}}{L_1} + \frac{6EI_{hc}}{L_2}$$

$$\frac{\partial M_B \times 10}{B} = -\frac{6 \times \frac{2}{3} \times 5 \times 18.5 \times 10^5}{5} - \frac{6 \times \frac{2}{3} \times 5 \times 15.625 \times 2.5}{5}$$

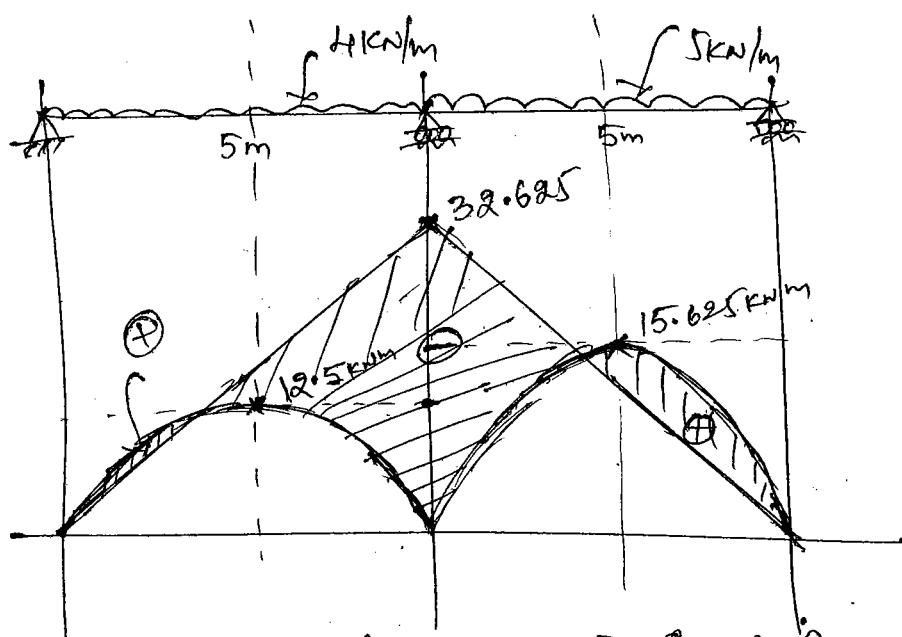
$$+ \frac{6 \times 18.5 \times 10^6 \times \frac{1}{1000}}{5} + \frac{6 \times 18.5 \times 10^6 \times \frac{1}{1000}}{5}$$

$$= -185 - 156.25 + 108 + 108$$

$$= -65.25 \Rightarrow M_B = \underline{\underline{-32.625 \text{ kNm}}}$$

-32.625 kNm

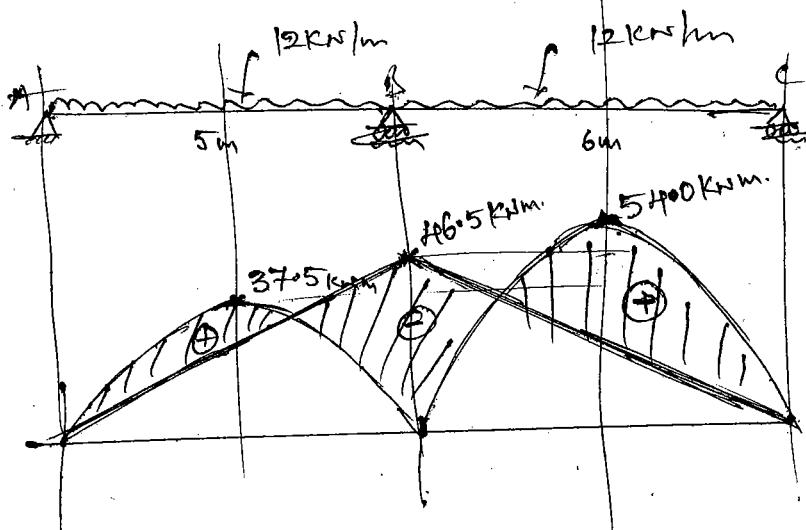
BMD:-



2016
Set 3
three moment theorem derivation?

Set 4
prob A continuous beam ABCD is SS over three spans. AB is 8m carrying UDL of 4kN/m, BC is 10m carrying UDL of 3kN/m and CD is 5m carrying UDL of 6kN/m. Find the moments over supports B and C. Draw BMD?

at A and C and continuous over support B with AB = 5m and BC = 6m. A UDL of 12 kN/m is acting over the beam. The moment of inertia is I throughout the span. Analyse the continuous beam and draw SFD and BMD?



Applying Clapeyron's eqn:-

$$M_A \left(\frac{L_1}{I_1} \right) + 2M_B \left(\frac{L_1}{I_1} + \frac{L_2}{I_2} \right) + M_C \left(\frac{L_2}{I_2} \right) = -\frac{6A_1a_1}{I_1L_1} - \frac{6A_2a_2}{I_2L_2} + \frac{6Eh_A}{L_1} + \frac{6Eh_C}{L_2}$$

\therefore EI is same throughout; $h_A = h_C = 0$

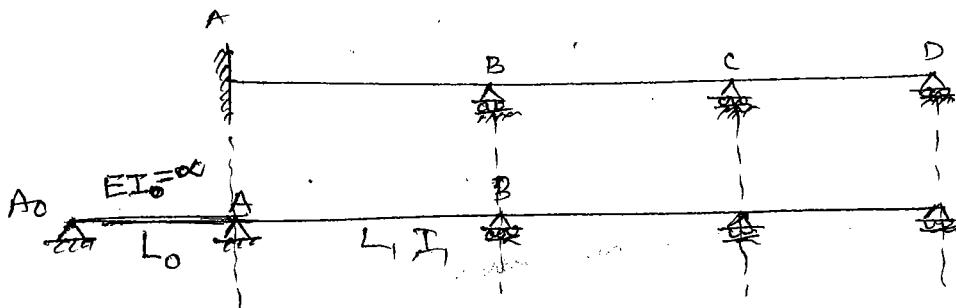
$$M_A = M_C = 0 \quad (\text{Simple end})$$

$$\therefore 2M_B (L_1 + L_2) = -\frac{6A_1a_1}{L_1} - \frac{6A_2a_2}{L_2}$$

$$2M_B (5+6) = -6 \times \frac{2}{3} \times \frac{1}{8} \times 37.5 \times 2.5 - 6 \times \frac{3}{8} \times \frac{1}{6} \times 54 \times 3$$

$$2M_B = -375 - 648 = -1023$$

$$\therefore M_B = 46.5 \text{ kNm}$$

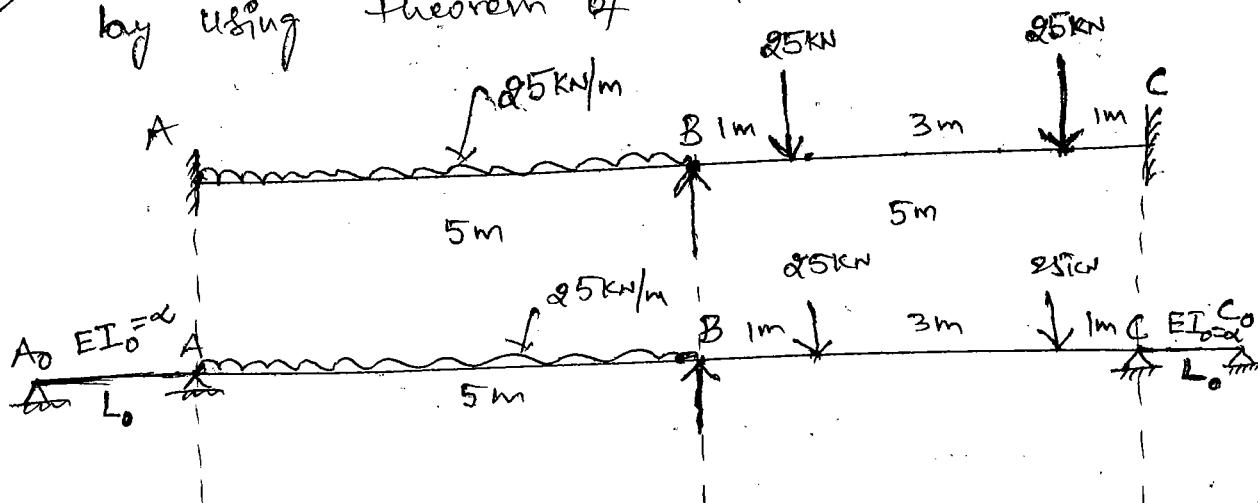


* Add an imaginary span $A A_0$ of length L_0 with infinite flexural Rigidity, $EI_0 = \alpha$

$$\therefore M_o\left(\frac{L_0}{\alpha}\right) + \alpha M_A\left(\frac{L_0}{\alpha} + \frac{L_1}{I_1}\right) + M_B\left(\frac{L_1}{I_1}\right) = -\frac{6A_0\alpha_0}{\alpha L_0} - \frac{6A_1\alpha_1}{I_1 L_1} + \frac{6Eh_b}{L_1}$$

$$\therefore \alpha M_A\left(\frac{L_1}{I_1}\right) + M_B\left(\frac{L_1}{I_1}\right) = -\frac{6A_1\alpha_1}{I_1 L_1} + \frac{6Eh_b}{L_1}$$

~~Supply section 15-16~~
prob:- Solve the continuous beam in below figure by using theorem of three moments?



Applying the three moment equation for A_0, A & B supports

$$\therefore M_o\left(\frac{L_0}{\alpha}\right) + \alpha M_A\left(\frac{L_0}{\alpha} + \frac{L_1}{I_1}\right) + M_B\left(\frac{L_1}{I_1}\right) = -\frac{6A_0\alpha_0}{\alpha L_0} - \frac{6A_1\alpha_1}{I_1 L_1} + \frac{6Eh_b}{L_1}$$

($\because h_b = 0$)

$$2M_A(L_1) + M_B(L_1) = -\frac{6A_1\alpha_1}{L_1} \Rightarrow (2M_A + M_B) \cancel{\frac{1}{L_1}} = -\frac{6A_1\alpha_1}{L_1}$$

$$2M_A + M_B = -156.25 \quad \rightarrow ①$$

$$-\frac{6}{L_1} \times \frac{2}{3} \times 8 \times \left(\frac{25 \times 5}{8}\right) \times 0.5$$

Applying three moment equation to A, B & C supports.

$$M_A \left(\frac{L_1}{I_1} \right) + \alpha M_B \left(\frac{L_1}{I_1} + \frac{L_2}{I_2} \right) + M_C \left(\frac{L_2}{I_2} \right) = -\frac{6Aa_1}{I_1 L_1} - \frac{6Aa_2}{I_2 L_2} + \frac{6Eh_a^{\gamma_0}}{L_1} + \frac{6Eh_c^{\gamma_0}}{L_2}$$

$\therefore h_a = h_c ; I_1 = I_2 = I$

$$M_A(5) + \alpha M_B(5+5) + M_C(5) = -\frac{6Aa_1}{L_1} - \frac{6Aa_2}{L_2}$$

$$M_A 5 + 20M_B + 5M_C = -\frac{6 \times \frac{8}{3} \times 5 \times (25 \times 5)^2}{82} \times \frac{2.5}{5}$$

$$= -6 \frac{(1 \times 25) + (25 \times 3)}{5} \times 2.5$$

$$5M_A + 20M_B + 5M_C = -781.25 - 300 = -1081.25$$

$$M_A + 4M_B + M_C = -216.25 \quad \text{--- (2)}$$

Applying three moment equation to B, C & C₀ supports

$$M_B \left(\frac{L_2}{I_2} \right) + \alpha M_C \left(\frac{L_2}{I_2} + \frac{L_0}{I_0} \right) + M_{C_0} \left(\frac{L_0}{I_0} \right) = -\frac{6A_2 a_2}{I_2 L_2} - \frac{6A_0 a_0}{I_0 L_0} + \frac{6Eh_b^{\gamma_0}}{L_2} + \frac{6Eh_{C_0}^{\gamma_0}}{L_0}$$

$\therefore h_b = 0 ; I_2 = I$

$$M_B(5) + \alpha M_C(5) = -\frac{6A_2 a_2}{L_2} = -300$$

$$M_B + \alpha M_C = -60 \quad \text{--- (3)}$$

$$= -156.25 \quad \text{--- (1)}$$

$$\underline{\alpha M_A + M_B + 0}$$

$$\cancel{\alpha M_A} + \cancel{2M_B} + \cancel{2M_C} = +216.25$$

$$2M_A + 8M_B + 2M_C = -432.5$$

$$6M_B = -216.25 \Rightarrow$$

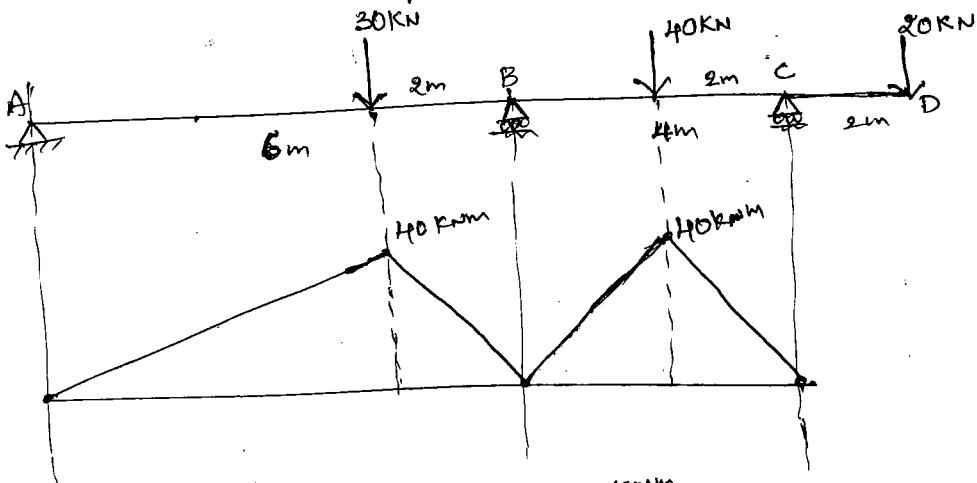
$$M_A = -60.104 \text{ KNM}$$

$$M_B = -36.042 \text{ KNM}$$

$$M_C = -11.98 \text{ KNM}$$

$$\therefore M_A = 60 \text{ KNM} ; M_B = -36 \text{ KNM} ; M_C = -12 \text{ KNM}$$

prob:- Analyse the continuous beam ABCD shown below, if support C settles down by 5mm. Take $E = 15 \text{ KN/mm}^2$. Moment of Inertia is constant throughout and is equal to $5 \times 10^9 \text{ mm}^4$



$$M_A = 0 \Rightarrow M_C = 20 \times 2 = -40 \text{ kNm}$$

Applying three moment equation,

$$0 + 2M_B(6+4) + M_C L_2 = \frac{-6A_1a_1}{L_1} - \frac{6A_2a_2}{L_2} \\ + \frac{6EIh_a}{L_1} + \frac{6EIh_c}{L_2}$$

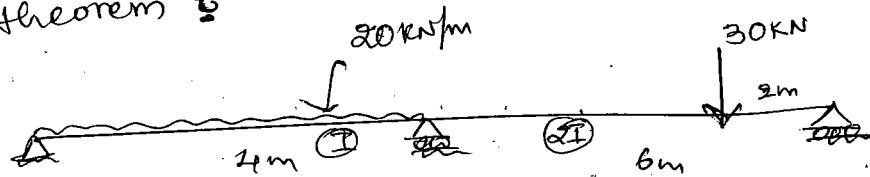
$$20M_B - 40 \times 4 = -6 \times \left(\frac{1}{2} \times 6 \times 40\right) \times \frac{(6+4)}{3} - 6 \times \left(\frac{1}{2} \times 4 \times 40\right) \times 2 \\ - \frac{6 \times 15 \times 10^6 \times 5 \times 10^{-3} \times 5 \times 10^{-3}}{4}$$

$$M_B = -52.125 \text{ kNm}$$

$$\therefore M_A = 0; M_B = -52.125 \text{ kNm}, M_C = -40 \text{ kNm}$$

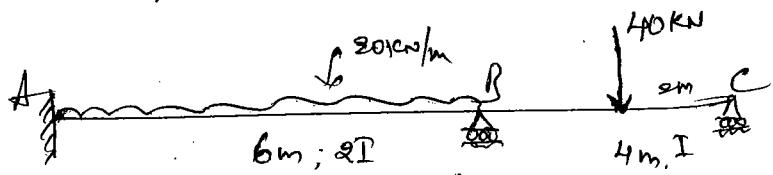
SSBKR: prob 8= Analyse the two span continuous beam shown below using three moment theorem

theorem :-



$$\text{Ans! } M_B = -34.286 \text{ kNm}$$

SPPU Analyse the Continuous beam shown by three moment equation? Draw BMD?

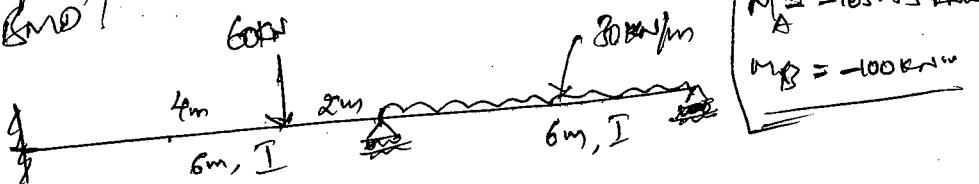


$$\text{Ans} \quad M_A = -69.6 \text{ kNm} ; \quad M_B = -40.8 \text{ kNm}$$

SPPU Analyse the continuous beam ABCDE shown below if support C fails by 8mm. Given, $E=200 \text{ kN/mm}^2$ $I = 0.8 \times 10 \text{ mm}^4$. Ans ($M_C = -10429 \text{ kNm}$)



SPPU Analyse the beam ABC shown below and draw BMD?



$$M_A = -103.13 \text{ kNm}$$

$$M_B = -100 \text{ kNm}$$

$$\frac{6A_1a_1}{L} = \left(\frac{8}{60}\right) \omega L^3$$

$$\frac{6A_2a_2}{L} = \left(\frac{4}{60}\right) \omega L^3$$

$$\frac{6A_1a_1}{L} = \left(\frac{4}{60}\right) \omega L^3$$

$$\frac{6A_2a_2}{L} = \left(\frac{8}{60}\right) \omega L^3$$

$$\left(\frac{5}{32}\right) \omega L^3$$

$$\left(\frac{5}{32}\right) \omega L^3$$

$$\frac{\omega}{4L} \left[\alpha_2^2 (\alpha L^2 - \alpha_2^2) - \alpha_1^2 (2L^2 - \alpha_1^2) \right]$$

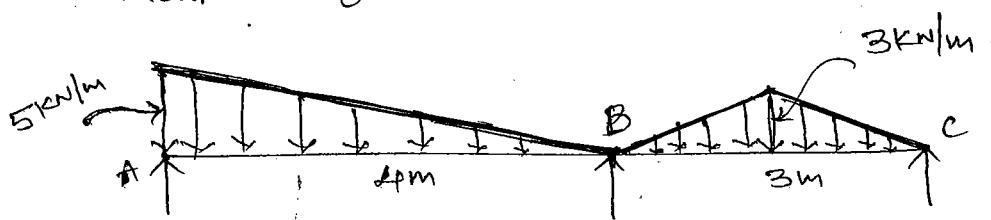
$$\frac{\omega}{4L} \left[\frac{\alpha_2^2 (2L^2 - \alpha_2^2)}{b_2^2} - \frac{\alpha_1^2 (2L^2 - \alpha_1^2)}{b_1^2} \right]$$

$$- \left[\frac{M}{L} \right] \left[3(L-\alpha)^2 - L^2 \right] \left(\frac{N}{L} \right) \left[(3\alpha^2 - L^2) \right]$$

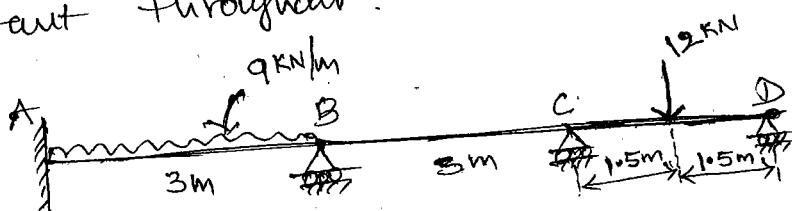
Continuous beams

Exercise Questions from T.S.TAM

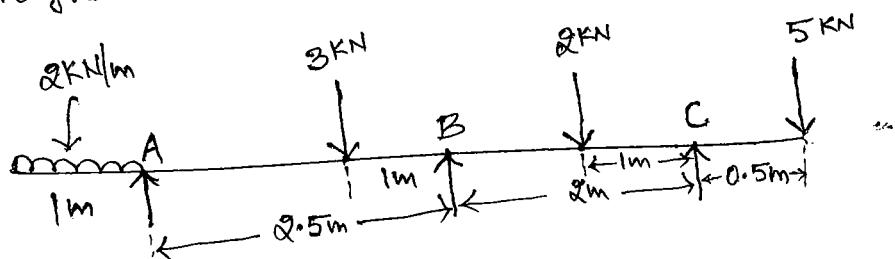
- ① Define a continuous beam?
- ② Derive Clapeyron's theorem of three moments including the effect of support settlements.
- ③ Analyse the beam shown below with constant EI using Clapeyron's three-moment equation?



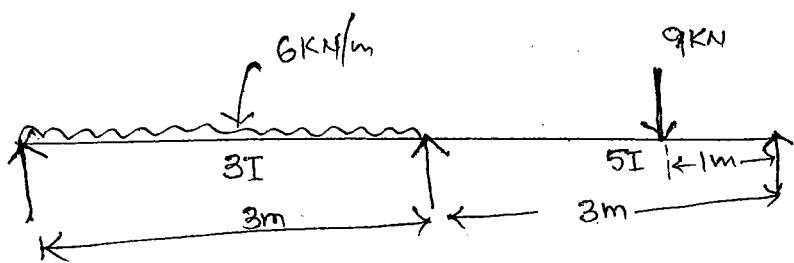
- ④ Using Clapeyron's theorem, solve the problem of continuous beam shown below. EI is constant throughout?



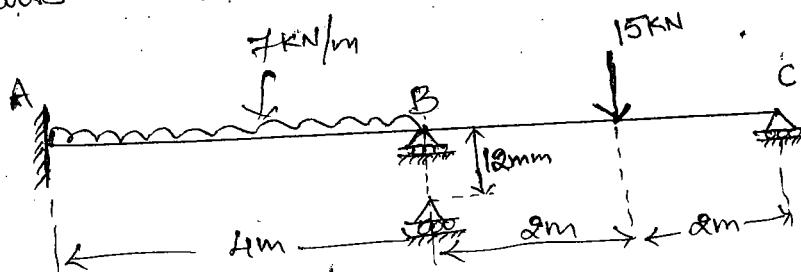
- ⑤ A continuous beam has overhangs on both sides as shown below. Apply three moment equation to determine the support moments. EI is constant throughout



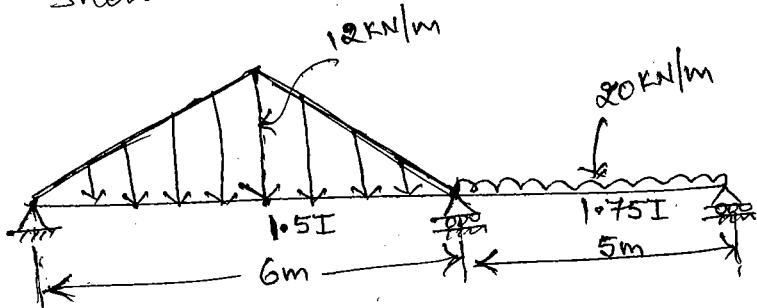
- ⑥ The moment of Inertia of a continuous beam is different for different spans as shown below. Find the reactions?



- ⑦ The support 'B' of a continuous beam shown below has settled by 18mm. Find out the moments at supports?



- ⑧ Draw the SFD and BMD for the continuous beam shown below?



SLOPE DEFLECTION METHOD

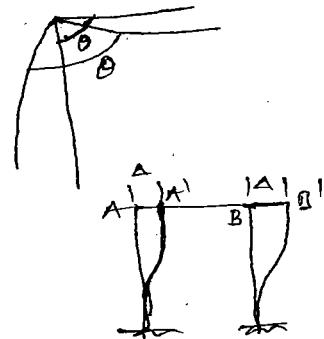
- Using this method basic unknowns like slopes and deflections of joints can be calculated.
- Moments at the ends of a member is first written in terms of unknown slopes and deflections of the end joints.
- Considering joint equilibrium conditions, a set of equations are formed and solution of these equations gives unknown slopes and deflections.
- Then end moments of individual members are determined.
- Since it involves solution of simultaneous equations, a problem with more than three unknowns is considered as difficult for hand calculation.
- Development of this method in the matrix form, has led to the "Stiffness matrix method".

Assumptions :- 1. All joints are rigid,

2. Distortions due to axial deformations are neglected

$$AA' = BB' = \Delta$$

3. Shear deformations are neglected



Sign Conventions:-

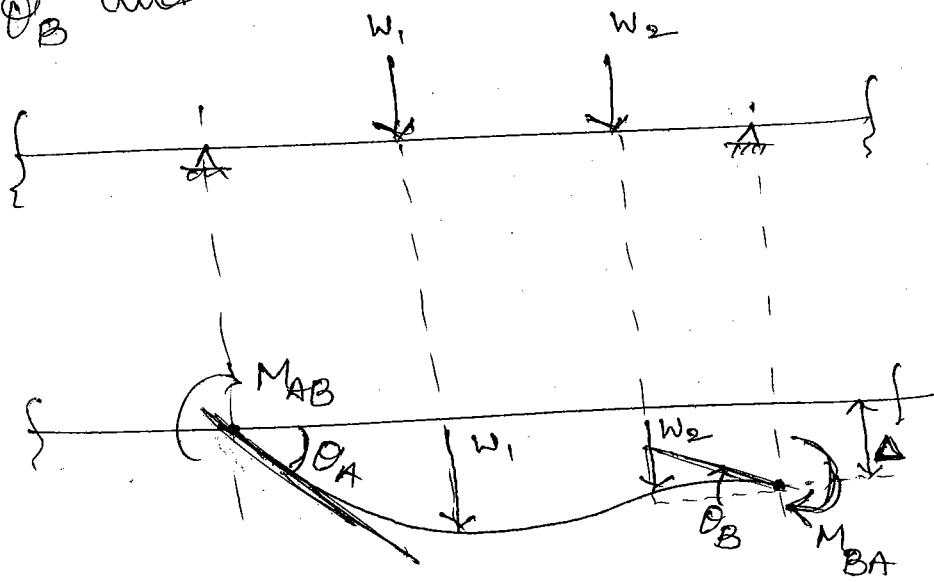
1. C.W moments are +ve; A.W moments are -ve

2. C.W Rotations " +ve; A.W Rotations " -ve

3. Settlement Δ is +ve if Right hand side support settles down
" " -ve " left "

Let AD, drawn below, be a member of a rigid structure. After loading it undergoes deformations. Final moments at end A and end B are M_{AB} and M_{BA} . Now our aim is to derive the relationship between these final end moments and their displacements

θ_A , θ_B and Δ .



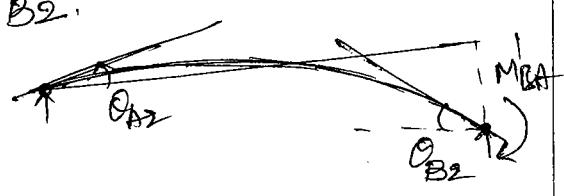
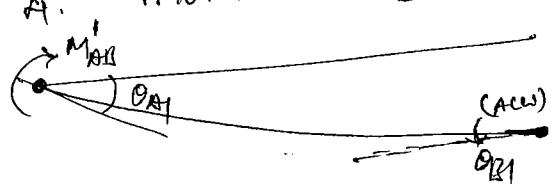
The solution & development of final end moments can be visualised in the following stages.

1. Due to given loadings end moments M_{FAB} , M_{FBA} develop without any rotations at ends. Hence called fixed end moments.

2. Settlement Δ takes place without any rotations at ends. This is similar to the settlement of supports in a fixed beam. Hence moments developed are $\frac{GEI\Delta}{L^2}$

3. moment M'_{AB} comes into play in SSB to cause end rotations θ_{A1}, θ_{B1} at A & B resp.

4. moment $M'_{BA} \Rightarrow \theta_{A2}, \theta_{B2}$.



$$\therefore \theta_{A1} = \frac{M'_{AEL}}{3EI} \quad \theta_{B1} = \frac{M'_{BEL}}{6EI}$$

$$\theta_{A2} = \frac{M'_{BAL}}{6EI} \quad \theta_{B2} = \frac{M'_{BAL}}{3EI}$$

$$\therefore \theta_A = \theta_{A1} - \theta_{A2} = \frac{M'_{AEL}}{3EI} - \frac{M'_{BAL}}{6EI}$$

$$\theta_B = \theta_{B2} - \theta_{B1} = \frac{M'_{BAL}}{3EI} - \frac{M'_{AEL}}{6EI}$$

$$2\theta_A + \theta_B = \frac{\cancel{2M'_{AEL}}}{3EI} - \cancel{\frac{M'_{BAL}}{3EI}} + \cancel{\frac{M'_{BAL}}{3EI}} - \cancel{\frac{M'_{AEL}}{6EI}}$$

$$= \cancel{3} \frac{M'_{AEL}}{2EI} \Rightarrow M'_{AB} = \frac{2EI}{L} (2\theta_A + \theta_B)$$

Why $M'_{BA} = \frac{2EI}{L} (2\theta_B + \theta_A)$

due to
Support Settlement

$$\therefore M_{AB} = M_{FAB} - \frac{6EI\Delta}{L^2} + M'_{AB} = M_{FAB} + M'_{AB} + M'_{AB}$$

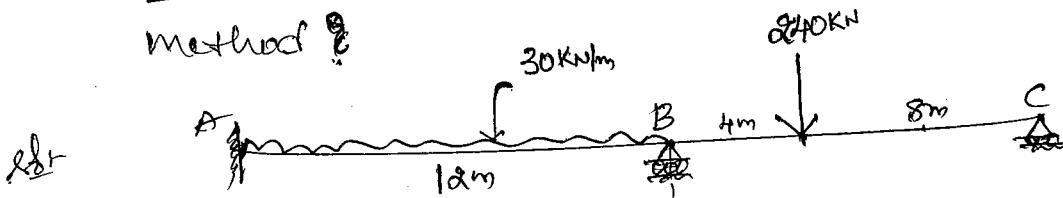
$$= \cancel{M_{FAB}} + \frac{2EI}{L} (2\theta_A + \theta_B) - \frac{6EI\Delta}{L^2}$$

$$M_{AB} = M_{FAB} + \frac{2EI}{L} \left[(2\theta_A + \theta_B) - \frac{3\Delta}{L} \right] \quad \text{--- (1)}$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L} \left[(2\theta_B + \theta_A) - \frac{3\Delta}{L} \right] \quad \text{--- (2)}$$

① & ② Are slope deflection Equations

over rollers at B and C. AB = BC = 12m. Beam carries a UDL of 30 kN/m over AB and a point load of 840 kN at a distance 4m from B on span BC. B has a settlement of 30mm. E = 2×10^5 N/mm², I = 2×10^9 mm⁴. Analyse the beam by slope deflection method.



Considering FEM_{AB} $M_{FAB} = -\frac{30 \times 12^2}{12} = -360 \text{ kNm}$

$$M_{FBA} = 360 \text{ kNm}$$

$$M_{FBC} = -\frac{240 \times 4 \times 8^2}{12^2} = -426.67 \text{ kNm}$$

$$M_{FCB} = +\frac{240 \times 8 \times 4^2}{12^2} = 213.33 \text{ kNm}$$

$$\begin{aligned} M_{AB} &= M_{FAB} + \frac{\alpha EI}{L} \left(\theta_A + \theta_B - \frac{3\Delta}{L} \right) \\ &= -360 + \frac{2 \times 10^5}{12} \left[\theta_A + \theta_B - \frac{3 \times 30}{120} \right] \\ &= -360 + \frac{2 \times 10^5}{3} \left[\theta_A + \theta_B - \frac{3}{400} \right] \end{aligned}$$

$$\begin{aligned} M_{BA} &= 360 + \frac{2 \times 10^5}{12} \left[\theta_B + \theta_A - \frac{3 \times 30}{120} \right] \\ &= 360 + \frac{2 \times 10^5}{3} \left[\theta_A + \theta_B - \frac{3}{400} \right] \end{aligned}$$

$$\begin{aligned} M_{BC} &= -426.67 + \frac{2 \times 10^5}{12} \left[\theta_B + \theta_C + \frac{3 \times 30}{120} \right] \\ &= -426.67 + \frac{2 \times 10^5}{3} \left[\theta_B + \theta_C + \frac{3}{400} \right] \end{aligned}$$

$$M_{CB} = 213.33 + \frac{2 \times 10^5}{3} \left[\theta_B + \theta_C + \frac{3}{400} \right]$$

Substituting, $\theta_A = 0$ and $\sum M_B = 0$,

$$M_{BA} + M_{BC} = 0 \rightarrow$$

$$360 + \frac{2 \times 10^5}{3} \left[\theta_A + 2\theta_B - \frac{3}{400} \right] + (-426.67) +$$

$$\frac{2 \times 10^5}{3} \left[2\theta_B + \theta_C + \frac{3}{400} \right] = 0$$

$$4\theta_B + \theta_C = \frac{66.67 \times 3}{2 \times 10^5} \rightarrow ①$$

$$M_C = 0 \Rightarrow M_{CB} = 0 \Rightarrow 13.33 + \frac{2 \times 10^5}{3} (\theta_B + 2\theta_C + \frac{3}{400}) = 0$$

$$\frac{2 \times 10^5}{3} (\theta_B + 2\theta_C) = -13.33$$

$$-\theta_B + \cancel{\theta_C} = + \frac{13.33 \times 3}{2 \times 10^5}$$

$$2(4\theta_B) + 2\cancel{\theta_C} = \frac{66.67 \times 6}{2 \times 10^5}$$

$$+ \theta_B = \frac{2540}{2 \times 10^5} \Rightarrow [1.814 \times 10^{-3}] = \theta_B$$

$$\theta_C = \frac{66.67 \times 3}{2 \times 10^5} - 4 \times 1.814 \times 10^{-3}$$

$$\boxed{\theta_C = -6.256 \times 10^{-3}}$$

$$\therefore M_{AB} = -360 + \frac{2 \times 10^5}{3} \left[(0 + 1.814 \times 10^{-3}) - \frac{3}{400} \right] = -739.067 \text{ KNM}$$

$$M_{BA} = 360 + \frac{2 \times 10^5}{3} \left[0 + 2 \times 1.814 \times 10^{-3} - \frac{3}{400} \right] = 101.867 \text{ KNM}$$

$$M_{BC} = -426.67 + \frac{2 \times 10^5}{3} \left[2 \times 1.814 \times 10^{-3} + (-6.256 \times 10^{-3}) + 7.5 \times 10^{-3} \right]$$
$$= -101.867 \text{ KNM}$$

$$\underline{M_{CB} = 0}$$

Set 2, 3, 4 \Rightarrow Similar problems with Data change

~~Diagram~~ of beam in below figure by slope deflection method?



writing FBM Δ $M_{FAB} = -\frac{20 \times 8^2}{12} = -106.67 \text{ KNm}$

$$M_{FBA} = \frac{20 \times 8^2}{12} = 106.67 \text{ KNm}$$

$$M_{FBC} = -\frac{100 \times 1 \times 5^2}{6^2} = -69.444 \text{ KNm}$$

$$M_{FCB} = 100 \times 1^2 \times 5 = 13.889 \text{ KNm}$$

~~edge deflection~~
~~EI constant~~ $M_{AB} = -106.67 + \frac{\partial EI}{L=8} (\theta_A + \theta_B - 0)$

$$M_{BA} = 106.67 + \frac{\partial EI}{8} (\theta_A + \theta_B - 0)$$

$$M_{BC} = -69.444 + \frac{\partial EI}{6} (\theta_B + \theta_C - 0)$$

$$M_{CB} = 13.889 + \frac{\partial EI}{6} (\theta_B + \theta_C - 0)$$

Substituting, $\theta_A = 0 ; \theta_C = 0$

Joint Equilibrium equation, $\sum M_B = 0 ; M_{BA} + M_{BC} = 0$

$$106.67 + \frac{\partial EI}{8} (0 + \theta_B) - 69.444 + \frac{\partial EI}{6} (2\theta_B + 0) = 0$$

$$37.226 + \frac{\partial EI \theta_B}{8} + \frac{\partial EI \theta_B}{6} = 0$$

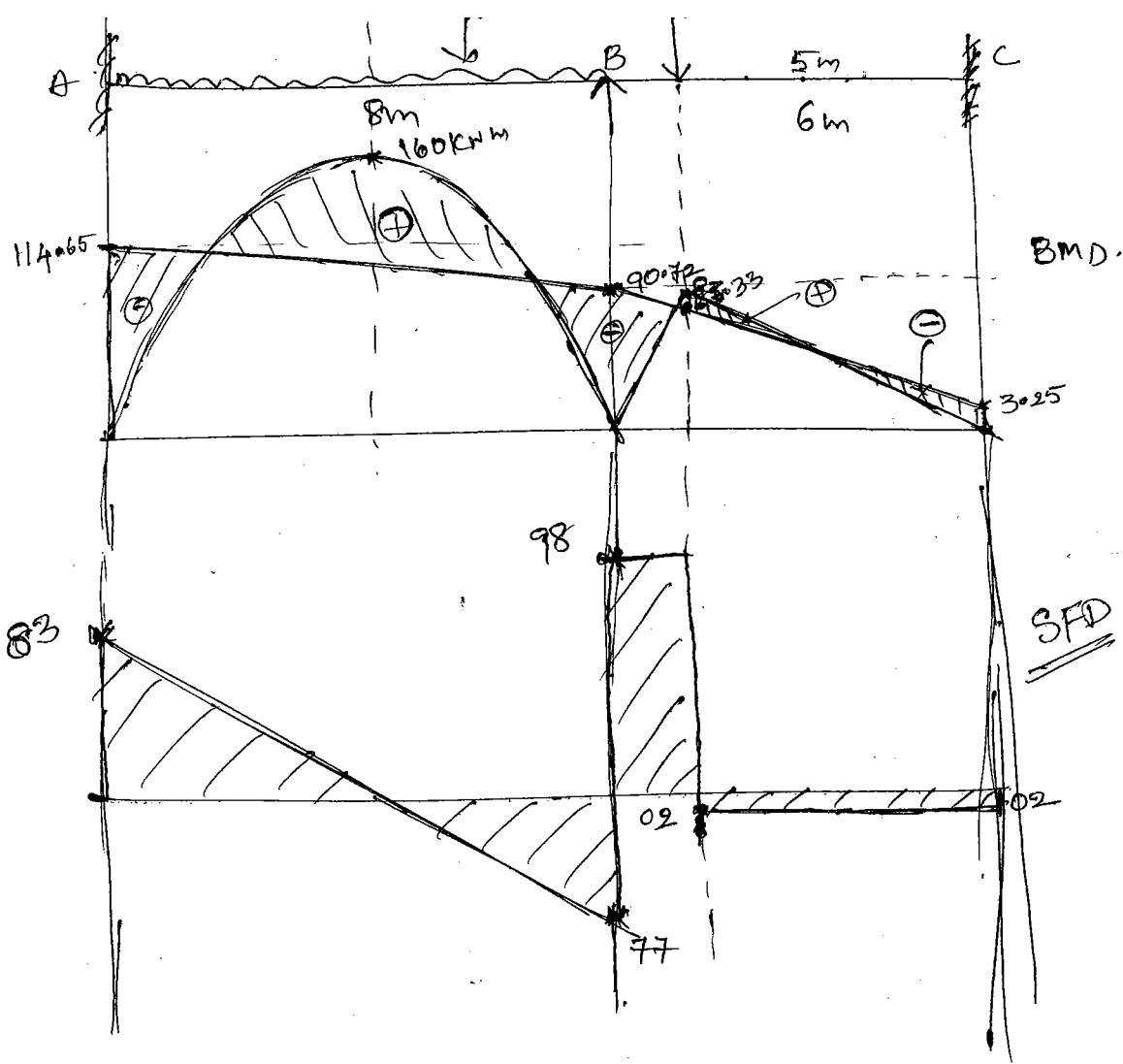
$$\theta_B \left(\frac{1}{8} + \frac{1}{6} \right) = -37.226 \quad \theta_B = -\frac{31.908}{EI}$$

$$\therefore M_{AB} = -106.67 + \frac{\partial EI}{8} \left(0 - \frac{31.908}{EI} \right) = -114.647 \text{ KNm}$$

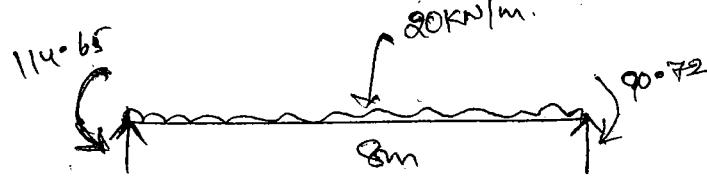
$$M_{BA} = 106.67 + \frac{\partial EI}{8} \left(0 - \frac{-31.908}{EI} \right) = 90.716 \text{ KNm}$$

$$M_{BC} = -69.444 + \frac{\partial EI}{6} \left(2 \times -\frac{31.908}{EI} \right) = -90.716 \text{ KNm}$$

$$M_{CB} = 13.889 + \frac{\partial EI}{6} \left(-\frac{31.908}{EI} \right) = 3.253 \text{ KNm}$$



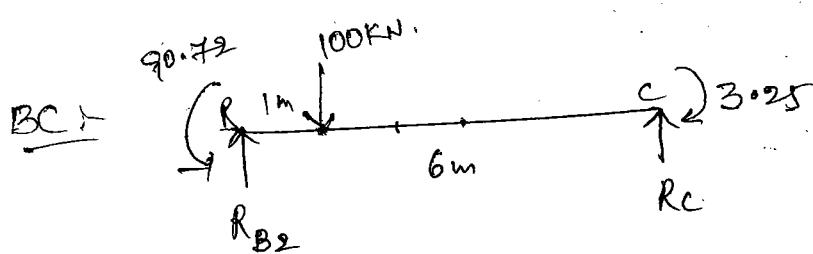
Shear force All



$$\sum M_A = 0$$

$$114.65 + R_B \times 8 - 90.72 - 20 \times 8 \times 4 = 0 \Rightarrow R_B = 77 \text{ kN}$$

$$R_A = 83 \text{ kN}$$



$$\sum M_B = 0$$

$$90.72 + R_C \times 6 - 3.25 - 100 = 0 \Rightarrow R_C = 2.0 \text{ kN}$$

$$R_{B2} = 98 \text{ kN}$$

~~major~~ ~~poor~~ ~~reduces~~

EI throughout its length. The end supports A and C are fixed and beam is continuous over middle support B. Span BC is uniformly loaded with 10 kN/m length while a concentrated vertical load of 100 kN acts at the mid-span of AB. Calculate the moments by slope deflection method.

Soln Assume Span AB = 6 m, BC = 4 m

$$\text{FEMM} \quad M_{FAB} = -\frac{100 \times 6}{8} = -75$$

$$M_{FBA} = 75$$

$$M_{FBC} = -\frac{10 \times 4^2}{12} = -13.33$$

$$M_{FCB} = 13.33$$

$$\text{SDEM} \quad M_{AB} = -75 + \frac{\alpha EI}{6} (\alpha \theta_A + \theta_B) = -75 + \frac{\alpha EI}{6} (2\theta_B)$$

$$M_{BA} = 75 + \frac{\alpha EI}{6} (\theta_A + 2\theta_B) = 75 + \frac{\alpha EI}{6} (2\theta_B)$$

$$M_{BC} = -13.33 + \frac{\alpha EI}{4} (2\theta_B + \theta_C) = -13.33 + \frac{\alpha EI}{4} (2\theta_B)$$

$$M_{CB} = 13.33 + \frac{\alpha EI}{4} (\theta_B + 2\theta_C) = 13.33 + \frac{\alpha EI}{4} (\theta_B)$$

Soln
By $\sum M_B = 0$; $M_{BA} + M_{AC} = 0$

$$75 + \frac{\alpha EI}{6} (2\theta_B) - 13.33 + \frac{\alpha EI}{4} (\theta_B) = 0$$

$$EI \theta_B \left(\frac{2}{3} + 1 \right) = -61.667$$

$$EI \theta_B = \frac{-61.667 \times 3}{5} = -37$$

$$\theta_B = -\frac{37}{EI}$$

$$M_{AB} = -75 + \frac{\alpha EI}{6} \left(-\frac{37}{EI} \right) = -87.333 \text{ KNM}$$

$$M_{BA} = 75 + \frac{\alpha EI}{6} \left(\alpha \times -\frac{37}{EI} \right) = 50.333 \text{ KNM}$$

$$M_{BC} = -13.33 + \frac{\alpha EI}{4} \left(\alpha \times -\frac{37}{EI} \right) = -50.333 \text{ KNM}$$

$$M_{CB} = 13.33 + \frac{\alpha EI}{4} \left(\theta_B - \frac{37}{EI} \right) = -5.167 \text{ KNM}$$

by slope deflection method? and draw BMD?



fixed end moments - $M_{FAB} = -\frac{80 \times 6^2}{8} = -60 \text{ kNm}$

$$M_{FBA} = 60 \text{ kNm}$$

$$M_{FCB} = -\frac{80 \times 4}{8} = -40 \text{ kNm}$$

$$M_{FCB} = 40 \text{ kNm}$$

slope deflection Equations $M_{AB} = -60 + \frac{\alpha EI(8I)}{6} (\theta_A + \theta_B - 0)$

$$M_{BA} = 60 + \frac{\alpha EI(8I)}{6} (\theta_A + \theta_B - 0) = -60 + \frac{\alpha EI}{3} \theta_B \quad (\because \theta_A = 0)$$

$$= 60 + \frac{4}{3} EI \theta_B$$

$$M_{BC} = -40 + \frac{\alpha EI}{4} (\theta_B + \theta_C - 0) = -40 + EI \theta_B + 0.5 EI \theta_C$$

$$M_{CB} = 40 + \frac{\alpha EI}{4} (\theta_B + \theta_C - 0) = 40 + 0.5 EI \theta_B + EI \theta_C$$

Equilibrium Equations: ① $\sum M_B = 0 ; M_{BA} + M_{BC} = 0$

$$60 + \frac{4}{3} EI \theta_B + (-40) + EI \theta_B + 0.5 EI \theta_C = 0$$

$$60 + \frac{4}{3} EI \theta_B + (-40) + EI \theta_B + 0.5 EI \theta_C = -20 \rightarrow ③$$

② $\sum M_c = 0 \Rightarrow M_{CB} + M_{CD} = 0$

$$\text{But, } M_{CD} = -40 \times 2 = -80 \quad \therefore M_{CB} = 80 \text{ kNm}$$

$$40 + 0.5 EI \theta_B + EI \theta_C = 80$$

$$0.5 EI \theta_B + EI \theta_C = 40 \rightarrow ④$$

$$EI \theta_B = -19.2$$

$$EI \theta_C = 49.6$$

Solving ③ & ④

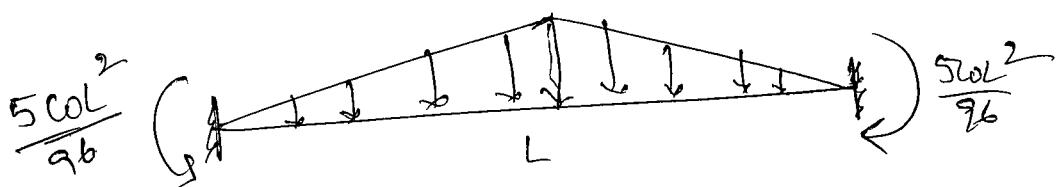
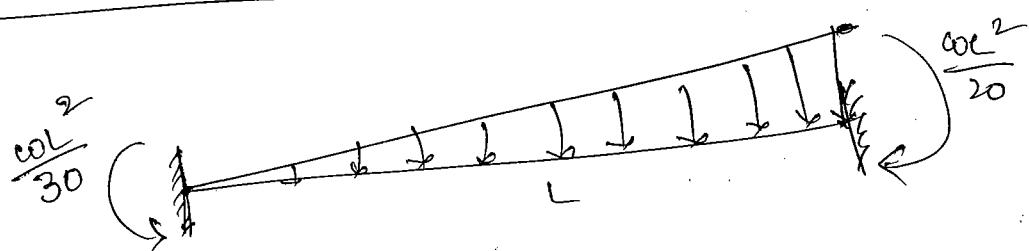
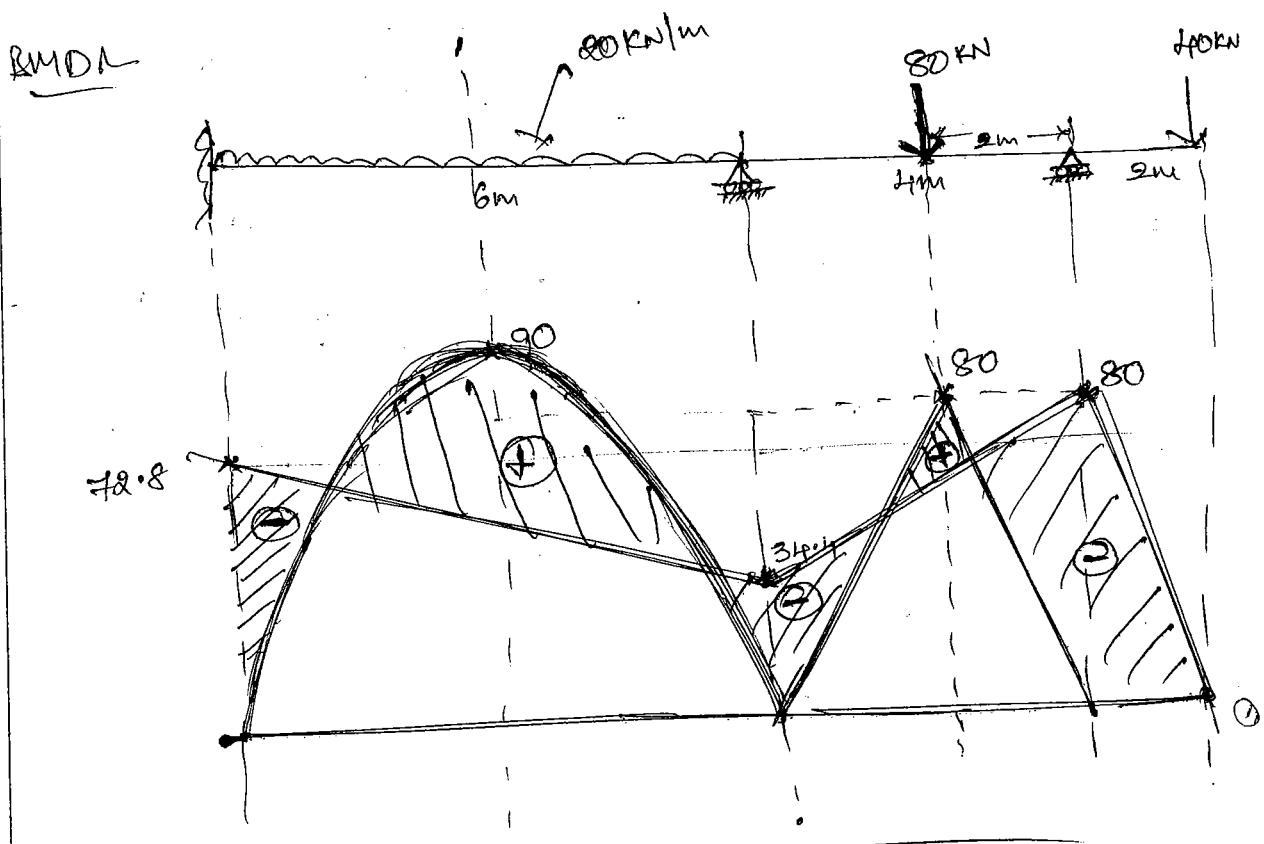
Substituting θ_B, θ_C in slope deflection equations we get fixed End moments

$$M_{AB} = -60 + \frac{8}{3}(-19.2) = -48.8 \text{ kNm}$$

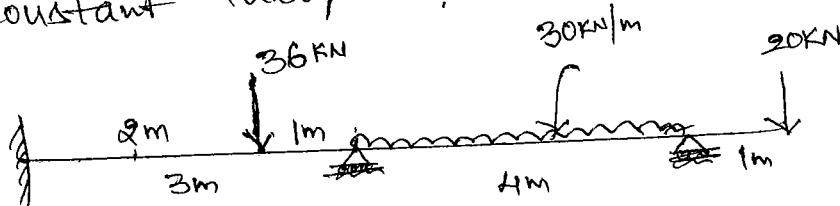
$$M_{BA} = 60 + \frac{4}{3}(-19.2) = 34.4 \text{ kNm}$$

$$M_{BC} = -40 + (-19.2) + 0.5(49.6) = -34.4 \text{ kNm}$$

$$M_{CB} = 80 \text{ kNm}$$

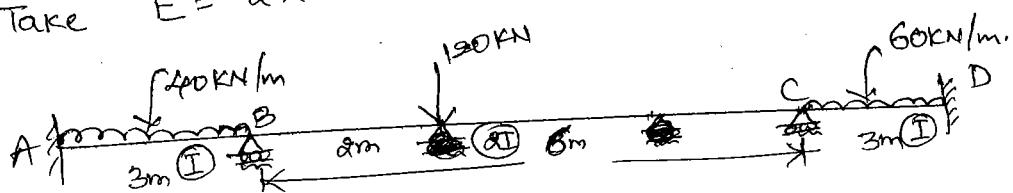


- ① Analyse the continuous beam shown in fig below by slope deflection method and draw bending moment diagram. flexural rigidity is constant throughout?



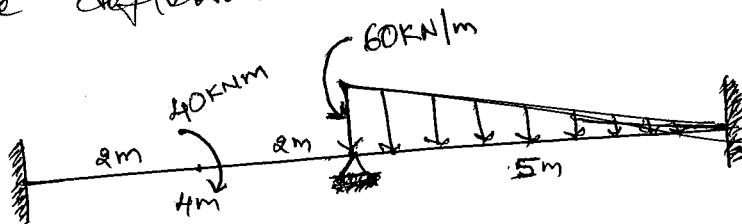
Ans: $M_{AB} = 2.88 \text{ kNm}$; $M_{BA} = 37.76 \text{ kNm}$; $M_{BC} = -M_{BC}$
 $M_{CB} = 20 \text{ kNm}$.

- ② Analyse the continuous beam ABCD shown in figure given below, if Support C sinks by 10mm. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 4 \times 10^7 \text{ mm}^4$.

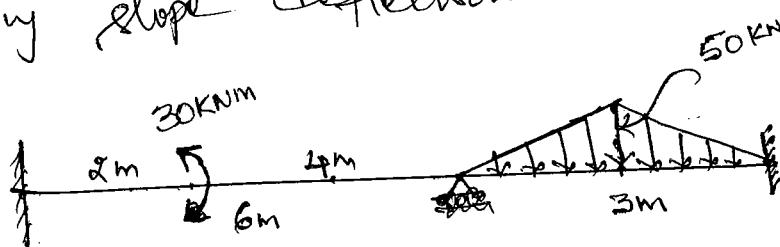


Ans:- $M_{AB} = -0.111 \text{ kNm}$ $M_{BA} = -M_{BC} = 89.777 \text{ kNm}$;
 $M_{CB} = -M_{CD} = 20.111 \text{ kNm}$; $M_{DC} = 88.666 \text{ kNm}$.

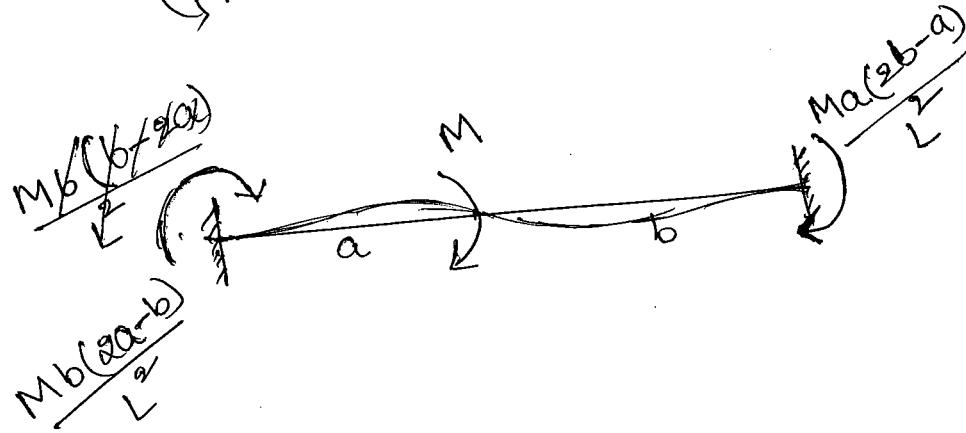
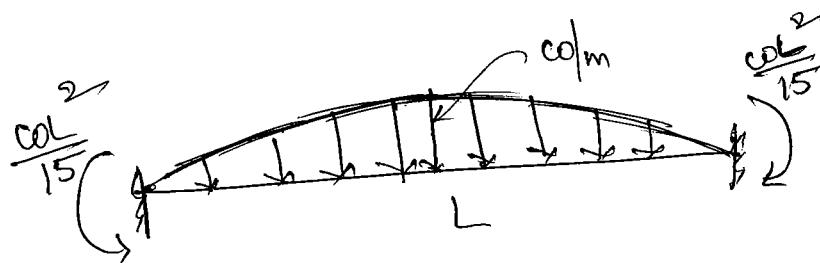
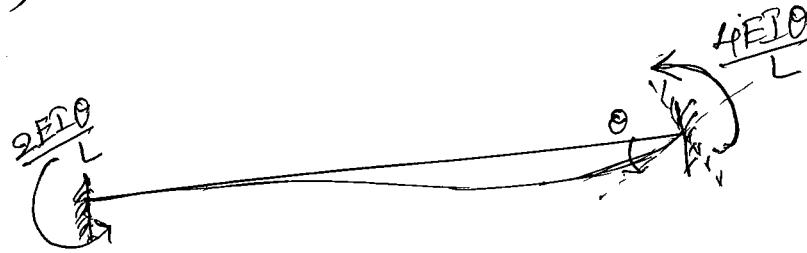
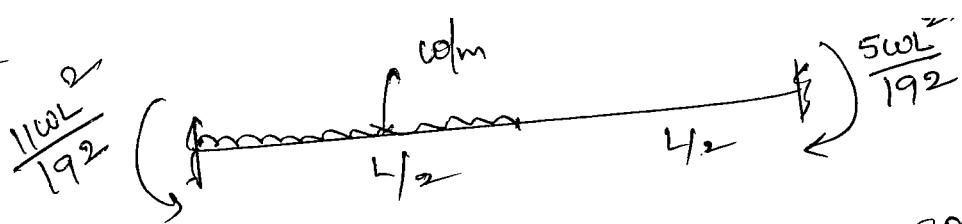
- ③ Analyse the continuous beam shown below using slope deflection method?



- ④ Analyse the continuous beam shown below using slope deflection method?



FEM's 1 -



Energy theorems

Introduction - S.E in linear elastic system - S.E due to axial load
BM & SF - Castigliano's first theorem - deflections of simple beams and pin jointed trusses.

Introduction:- Force is an action that tends to change the position of the body, either in a state of rest or in a state of motion, to which it is applied.

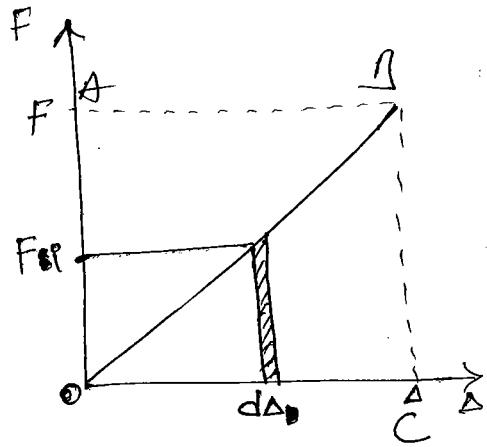
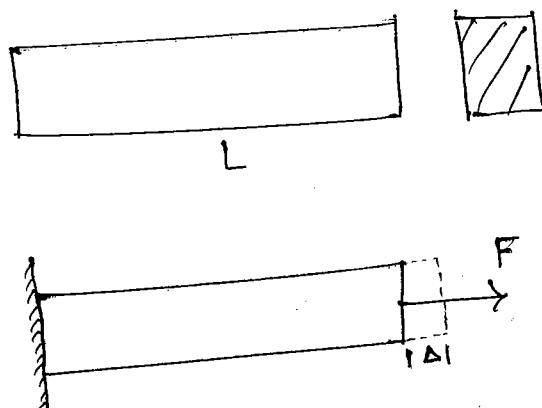
- * When the point of application of an active force moves, then the force is said to do work, (W); It is defined as the product of the force and the linear displacement of its point of application in the direction of force.
Capacity to do work is called the "Energy".
- * Work done by the force due to deformation of the body is stored up in the body as a particular form of potential energy, which is called as "Strain Energy".
- * In solid deformable bodies such as bars, beams and so on the product of stresses and areas is the force, and the deformations are the distances. the product of these quantities is the internal work done in a body by externally applied forces. The internal work is stored in the body as Internal Elastic Energy & "Elastic Strain Energy"

Strain Energy in Linear Elastic System

- * Strain Eng. is normally recoverable when the component or system returns to its original undeformed state provided that the material of the component is stressed within its elastic range.

e.g. mechanism of stress, strain, theory

Let us consider a bar



$$k_b = F/d\Delta \Rightarrow \text{total work done} = \int_0^A F_i d\Delta$$

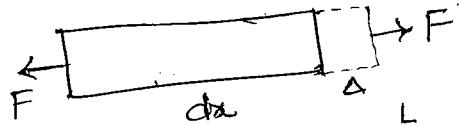
$$\therefore U = \int_0^A F_i d\Delta = \frac{1}{2} F \Delta \text{ (Total Area)}$$

Area under the $F-\Delta$ curve OBC is called strain energy

Area OAB, above the $F-\Delta$ curve is called complementary strain energy.

Expression for Axial load

$$\Delta = \frac{F dx}{AE}$$



$$\therefore U = \int_0^L \frac{F^2 dx}{2AE} = \int_0^L \left(\frac{1}{2}\right) (\Delta)(F) = \int_0^L \left(\frac{1}{2} \left(\frac{F dx}{AE}\right)\right) F$$

If F is constant,

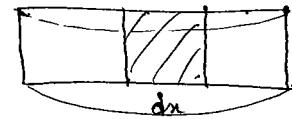
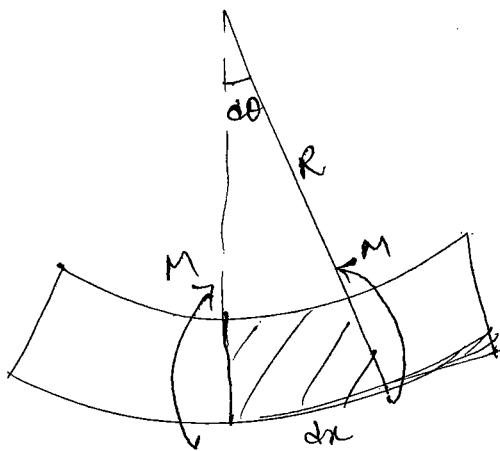
$$\therefore U = \frac{F^2}{2AE} (\Delta)_0^L = \frac{F^2 L}{2AE}$$

If there are many members, like that of a truss,

$$U_F = \frac{\sum F L}{2AE}$$

Strain Energy due to Bending:-

(2) - V



$$\frac{dx}{da} = \frac{R d\theta}{}$$

$$d\theta = \frac{dx}{R} = \frac{M dx}{EI}$$

$$\text{from } \frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\frac{1}{R} = \frac{M}{EI}$$

$$\therefore \text{for } 0 \text{ to } L \quad U = \int_0^L \left(\frac{1}{2}\right) M \cdot da$$

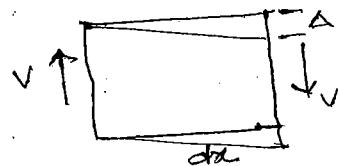
$$U = \int_0^L \frac{1}{2} \times M \times \frac{M dx}{EI}$$

$$U = \int_0^L \frac{M^2 dx}{2EI}$$

Strain Energy stored due to Shear:-

Shear Strain $\gamma = \frac{v}{c}$

$$\begin{aligned} \text{Shear Strain } \gamma &= \frac{v}{c} \\ &= \frac{V}{A_c c} \end{aligned}$$



$$\delta_v = \frac{V dx}{A_c c}$$

$$U_v = \int_0^L \left(\frac{1}{2}\right) v \times \Delta = \int_0^L \frac{1}{2} \times v \times \frac{V dx}{A_c c}$$

$$U_v = \int_0^L \frac{V^2 dx}{2A_c c}$$

Strain eng. due to torsion

from torsion eqn. $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$

$$\therefore \frac{TL}{GJ} = \theta \Rightarrow \frac{T dx}{GJ} = d\theta.$$

$$U = \int_0^L \left(\frac{1}{2}\right) (T) d\theta$$

$$= \int_0^L \frac{1}{2} (T) \left(\frac{T dx}{GJ} \right)$$

$$U_T = \int_0^L \frac{T^2 dx}{2GJ}$$

Castigliano's Theorems:-

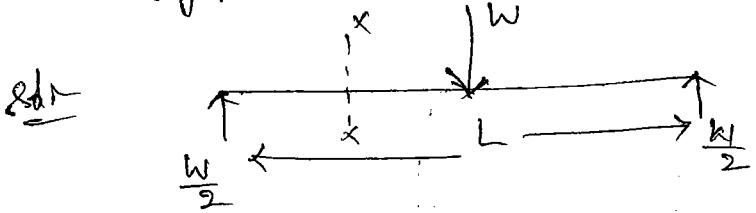
First Theorem- In a linearly elastic structure subjected to a system of external forces in static equilibrium, the partial derivative of the strain energy expressed in terms of displacement only, with respect to any displacement Δ_i at coordinate i is equal to the force F_i at that coordinate

$$\boxed{\frac{\partial U}{\partial \Delta_i} = F_i}$$

Second Theorem- In a linearly elastic structure in static equilibrium, the partial derivative of strain energy expressed in terms of forces with respect to any force F_i , at coordinate i is equal to the displacement Δ_i at that coordinate.

$$\boxed{\frac{\partial U}{\partial F_i} = \Delta_i}$$

Q3 A beam of length L carries a point load W at the centre. Find the deflection using energy theorem? (4M)



$$M_x = \frac{Wx}{2} \quad 0 \leq x \leq \frac{L}{2}$$

$$U = \int \frac{M_x^2 dx}{2EI} = \frac{1}{2} \int_0^{\frac{L}{2}} \frac{M_x^2 dx}{2EI}$$

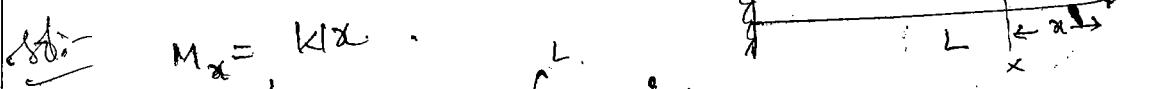
$$= \frac{1}{2} \int_0^{\frac{L}{2}} \frac{\left(\frac{Wx}{2}\right)^2 dx}{2EI} = \frac{W^2}{4EI} \cdot \left(\frac{x^3}{3}\right) \Big|_0^{\frac{L}{2}}$$

$$= \frac{W^2}{4EI} \cdot \frac{1}{3} \times \frac{L^3}{8} = \frac{WL^3}{96EI}$$

$$W \cdot D = \frac{1}{2} k l \delta = \text{Strain Eng. } U$$

$$= \frac{1}{2} k l f = \frac{W^2 L^3}{96EI} \Rightarrow S = \boxed{S = \frac{WL^3}{48EI}}$$

prob- A cantilever beam of length L carries a point load W at the free end. Find the deflection at free end using energy theorem? (4M)



$$M_x = kx$$

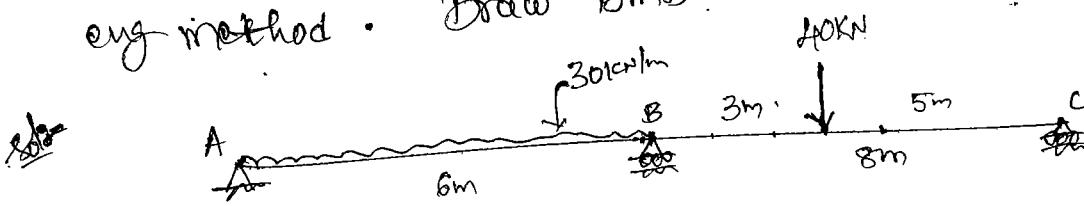
$$U = \int_0^L \frac{M_x^2 dx}{2EI} = \int_0^L \frac{(kx)^2 dx}{2EI}$$

$$= \frac{k^2}{2EI} \times \left(\frac{x^3}{3}\right) \Big|_0^L = \frac{W^2 L^3}{6EI}$$

$$\text{Work done} = \frac{1}{2} W \times S = U = \frac{WL^3}{6EI}$$

$$S = \boxed{S = \frac{WL^3}{3EI}}$$

Prob.
Span AB is 6m and BC is 8m. Span AB is carrying a UDL of 30 kN/m and span BC carries a load of 40 kN at a distance 3m from B. Use strain eng method. Draw BMD?



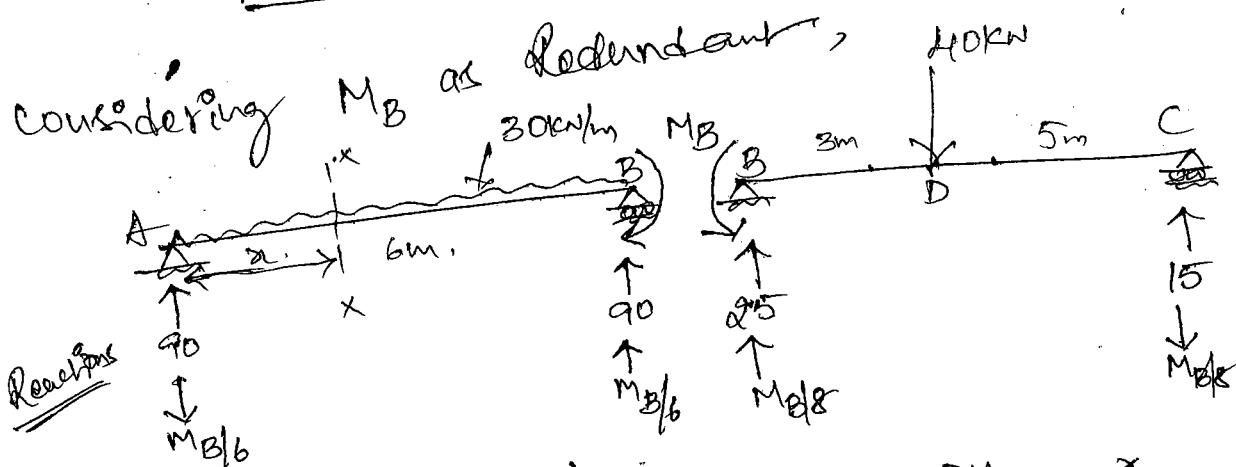
from Castigliano's Second theorem, $\frac{\partial U}{\partial F} = \Delta$

$$U_m = \int \frac{M_n^2 dx}{2EI}$$

$$\therefore \frac{\partial}{\partial F} \int \frac{M_n^2 dx}{2EI} = \Delta \quad | \quad \text{from the theorem of least work,}$$

"In a statically indeterminate structure, if there are no support settlements and no change of temperature, the redundants must be such as to make the strain eng minimum".

$$\therefore \boxed{\frac{\partial U}{\partial R} = 0} \quad \sum \int M_a \left(\frac{\partial M_x}{\partial R} \right) \frac{dx}{EI} = 0$$



$$\underline{AB} \quad M_{ax} = \left(Q_0x - \frac{M_B x}{6} - \frac{30x^2}{2} \right)$$

$$\underline{BD} \quad M_{ax} = 25x + \frac{M_B x}{8} - M_B$$

$$\underline{CD} \quad M_{ax} = 15x - \frac{M_B x}{8}$$

$$0 \leq x \leq 6 \quad \frac{\partial M_{ax}}{\partial M_B} = -\frac{x}{6}$$

$$0 \leq x \leq 3 \quad \frac{\partial M_{ax}}{\partial M_B} = \frac{x}{8} - 1$$

$$0 \leq x \leq 5 \quad \frac{\partial M_{ax}}{\partial M_B} = \left(-\frac{x}{8} \right)$$

(A) → V

$$\frac{\partial U}{\partial M_B} = 0 \Rightarrow \int_{0}^{6m} \left(\frac{\partial U}{\partial M_B} \right) \frac{dx}{EI} = 0$$

$$\frac{1}{EI} \int_0^6 \left(90x - \frac{M_B x}{6} - 15x^2 \right) \left(\frac{-x}{6} \right) dx +$$

$$\frac{1}{EI} \int_0^3 \left(25x + \frac{M_B x}{8} - M_B \right) \left(\frac{x}{8} - 1 \right) dx +$$

$$\frac{1}{EI} \int_0^5 \left(15x - \frac{M_B x}{8} \right) \left(\frac{-x}{8} \right) dx = 0$$

$$\frac{1}{EI} \left[\int_0^6 \left(-15x^2 + \frac{M_B x^2}{36} + 2.5x^3 \right) dx + \int_0^3 \left(3.125x^2 + \frac{M_B x^2}{64} \right) \right]$$

$$- M_B x - 2.5x - \frac{M_B x}{8} + M_B \right] dx +$$

$$\int_0^5 \left(-1.875x^2 + \frac{M_B x^2}{64} \right) dx = 0$$

$$-15 \left(\frac{x^3}{3} \right)_0^6 + \frac{M_B}{36} \left(\frac{x^3}{3} \right)_0^6 + 2.5 \left(\frac{x^4}{4} \right)_0^6 + 3.125 \left(\frac{x^3}{3} \right)_0^3$$

$$+ \frac{M_B}{64} \left(\frac{x^3}{3} \right)_0^3 - \frac{M_B}{4} \left(\frac{x^2}{2} \right)_0^3 - 25 \left(\frac{x^2}{2} \right)_0^3 + M_B \left(x \right)_0^3$$

$$+ (-1.875) \left(\frac{x^3}{3} \right)_0^5 + \frac{M_B}{64} \left(\frac{x^3}{3} \right)_0^5 = 0$$

$$-1080 + 2M_B + 810 + 28.0125 + 0.141M_B - 10.125M_B$$

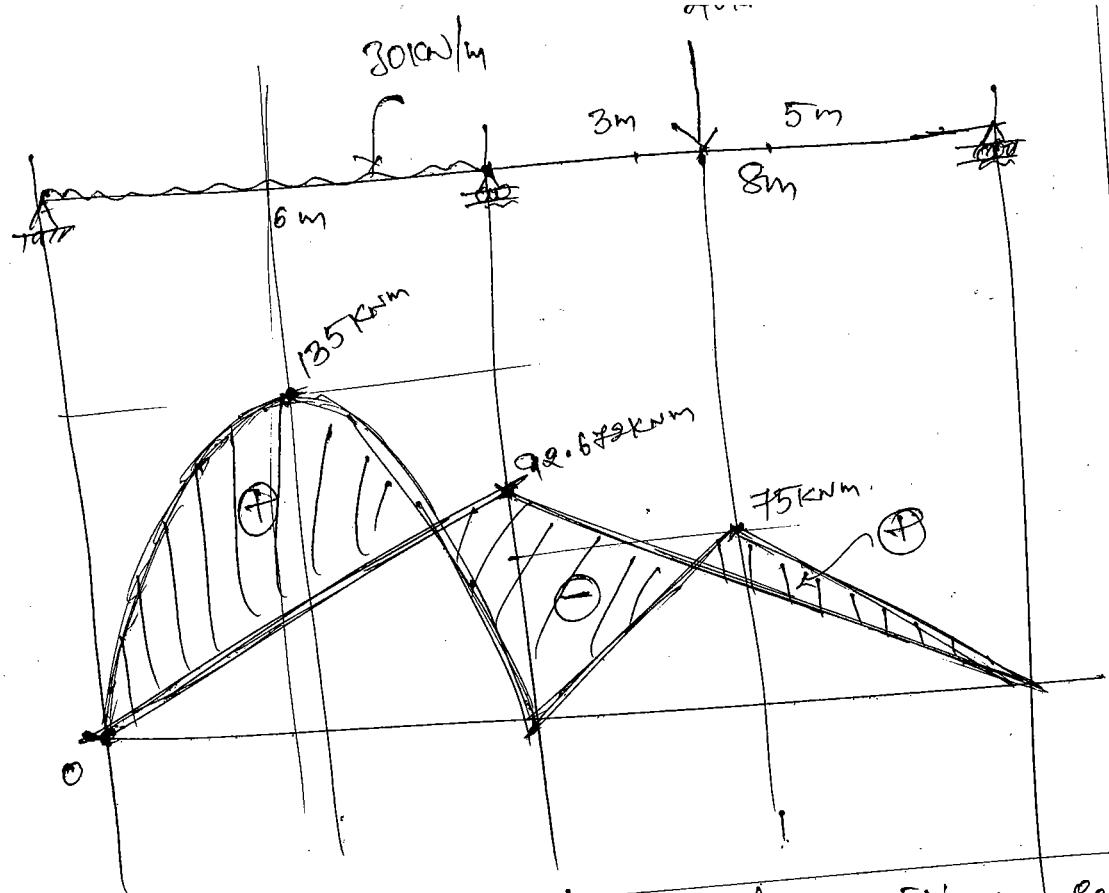
$$-112.5 + 3M_B - 78.0125 + 0.651M_B = 0$$

$$4.664M_B = 432.5 \Rightarrow \underline{\underline{M_B = 92.672 \text{ kNm}}}$$

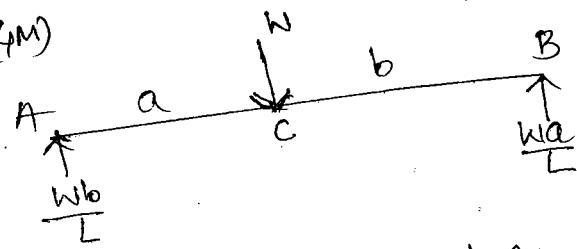
$$M_A = 0$$

$$M_C = 0$$

$$\underline{\underline{M_B = 92.672 \text{ kNm} (\text{Hog})}}$$



prob A Simply Supported beam of length 'L' carries an eccentric load w . Find the deflection using energy theorem? (4M)



$$U = \int_0^a \frac{(wbx)^2}{2EI} dx + \int_0^b \frac{(wax)^2}{2EI} dx$$

$$M_a = \frac{wbx}{L}$$

$$M_b = \frac{wax}{L}$$

$$= \left(\frac{wb}{L}\right)^2 \left(\frac{x^3}{3}\right)_0^a \left(\frac{1}{2EI}\right) + \left(\frac{wa}{L}\right)^2 \frac{1}{2EI} \times \left(\frac{x^3}{3}\right)_0^b$$

$$= \frac{w^2 b^2}{L^2} \frac{a^3}{3} \times \frac{1}{2EI} + \frac{w^2 a^2}{L^2} \times \frac{b^3}{3 \times 2EI}$$

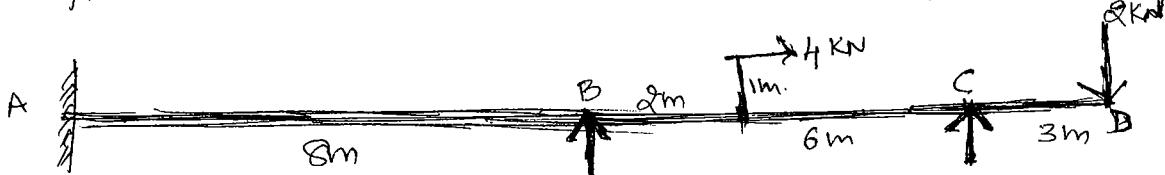
$$W.D = \frac{1}{2} W \times 8 = \cancel{W \times 8} \quad \frac{w^2 b^2 a^3}{8L^2 EI} + \frac{w^2 a^2 b^3}{3 \times 8L^2 EI}$$

$$\delta = \frac{w^2 b^2 a^3}{3L^2 EI} + \frac{w^2 a^2 b^3}{3L^2 EI} = \frac{w^2 b^2}{3L^2 EI} (a+b)$$

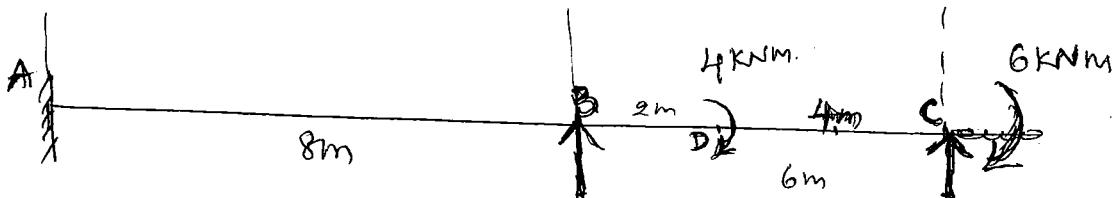
$$\delta = \frac{w^2 b^2}{3L^2 EI}$$

$$\therefore a+b=L$$

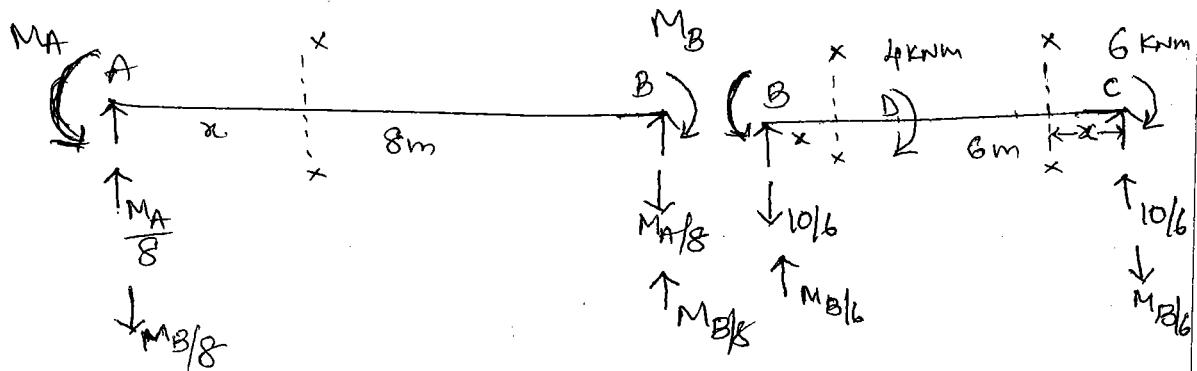
Q5 → A continuous beam of constant moment of inertia is loaded as shown below. Find the support moments. Use Strain Energy Method.



Sol:- modified Beam,



considering M_A & M_B as Redundants,



Span AB $M_x = \frac{M_A x}{8} - \frac{M_B x}{8} - M_A$ 0 to 8 $\frac{\partial M_x}{\partial M_A} = \frac{x}{8} - 1$
 $\frac{\partial M_x}{\partial M_B} = -\frac{x}{8}$

Span BC :- $M_x = \frac{M_B x}{6} - \frac{10x}{6} - M_B$ 0 to 6m $\frac{\partial M_x}{\partial M_A} = 0$
 $\frac{\partial M_x}{\partial M_B} = \frac{x}{6} - 1$

CD :- $M_x = \frac{10x}{6} - \frac{M_B x}{6} - 6$ 0 to 4m $\frac{\partial M_x}{\partial M_A} = 0$
 $\frac{\partial M_x}{\partial M_B} = -\frac{x}{6}$

From theorem of least work:- (Castigliano's 2nd theorem)

$$\frac{\partial U}{\partial M_A} = 0 ; \quad \frac{\partial U}{\partial M_B} = 0$$

$$\frac{\partial U}{\partial M_A} = 0 \Rightarrow \sum \int M_A \left(\frac{\partial M_A}{\partial M_A} \right) \frac{dx}{EI} = 0$$

$$\frac{1}{EI} \left[\int_0^8 \left(\frac{M_A x}{8} - \frac{M_B x}{8} - M_A \right) \left(\frac{x}{8} - 1 \right) dx + \int_0^4 \left(\frac{M_B x}{6} - \frac{10x}{6} - M_B \right) (0) dx \right. \\ \left. + \int_0^4 () \cdot 0 dx \right] = 0$$

$$\therefore \left(\frac{M_A}{8} \right) \left(\frac{x^2}{2} \right)_0^8 - \frac{M_B}{8} \left(\frac{x^2}{2} \right)_0^8 + M_A$$

$$\left[\int_0^8 \left(\frac{M_A x^2}{64} - \frac{M_B x^2}{64} - \frac{M_A x}{8} - \frac{M_A x}{8} + \frac{M_B x}{8} + M_A \right) dx = 0 \right]$$

$$\frac{M_A}{64} \left(\frac{x^3}{3} \right)_0^8 - \frac{M_B}{64} \left(\frac{x^3}{3} \right)_0^8 - \frac{M_A}{8} \left(\frac{x^2}{2} \right)_0^8 - \frac{M_A}{8} \left(\frac{x^2}{2} \right)_0^8 + \frac{M_B}{8} \left(\frac{x^2}{2} \right)_0^8 + M_A \left(x \right)_0^8 = 0$$

$$\frac{M_A}{64} \times \frac{8^3}{3} - \frac{M_B}{64} \times \frac{8^3}{3} - \frac{M_A}{8} \times \frac{8^2}{2} - \frac{M_A}{8} \times \frac{8^2}{2} + \frac{M_B}{8} \times \frac{8^2}{2} + M_A (8) = 0$$

$$\left(M_A \left(\frac{8}{3} \right) - M_A \left(\frac{8}{2} \right) - M_A \left(\frac{8}{2} \right) + M_A (8) \right) - M_B \times \frac{8}{3} + M_B \times \frac{8}{2} = 0$$

$$M_A \left(\frac{8}{3} - \frac{8}{2} + 8 \right) - M_B \left(\frac{8}{3} + \frac{8}{2} \right) = 0$$

$$M_A \times \frac{8}{3} - M_B \times \frac{(16+24)}{6} = 0$$

$$M_A \times \frac{8}{3} - M_B \times \frac{40}{6} = 0$$

$$\boxed{8M_A + 10M_B = 0} \quad \text{--- (1)}$$

$$\frac{\partial U}{\partial M_B} = 0 \Rightarrow \sum \int M_B \left(\frac{\partial M_B}{\partial M_B} \right) \frac{dx}{EI} = 0$$

$$\frac{1}{EI} \left[\int_0^8 \left(\frac{M_A x}{8} - \frac{M_B x}{8} - M_A \right) \left(\frac{-x}{8} \right) dx + \int_0^2 \left(\frac{M_B x}{6} - \frac{10x}{6} - M_B \right) \left(\frac{x}{6} - 1 \right) dx \right]$$

$$+ \int_0^4 \left(\frac{10x}{6} - \frac{M_B x}{6} - 6 \right) \left(\frac{-x}{6} \right) dx = 0$$

(6)-V

$$\int_0^8 \left(-\frac{M_A x^2}{64} + \frac{M_B x^2}{64} + \frac{M_A x}{8} \right) dx + \int_0^2 \left(\frac{M_B x^2}{36} - \frac{10x^2}{36} - \frac{M_B x}{6} \right. \\ \left. - \frac{M_B x}{6} + \frac{10x}{6} + M_B \right) dx + \int_0^4 \left(-\frac{10x^2}{36} + \frac{M_B x^2}{36} + x \right) dx = 0$$

$$-\frac{M_A}{64} \left(\frac{x^3}{3}\right)_0^8 + \frac{M_B}{64} \left(\frac{x^3}{3}\right)_0^8 + \frac{M_A}{8} \left(\frac{x^2}{2}\right)_0^8 + \frac{M_B}{36} \left(\frac{x^3}{3}\right)_0^2 - \frac{10}{36} \left(\frac{x^3}{3}\right)_0^2$$

$$-\frac{M_B}{3} \left(\frac{x^2}{2}\right)_0^2 + \frac{10}{6} \left(\frac{x^2}{2}\right)_0^2 + M_B(x)_0^2 + \left(\frac{10}{36} \left(\frac{x^3}{3}\right)\right)_0^4$$

$$+ \left(\frac{M_B}{36} \right) \left(\frac{x^3}{3} \right)_0^4 + \left(\frac{x^2}{2} \right)_0^4 = 0$$

$$-\underline{M_A \left(\frac{8}{3}\right)} + \underline{M_B \left(\frac{8}{3}\right)} + \underline{M_A (4)} + \underline{M_B \times \frac{x}{27}} - \frac{20}{27} - \underline{M_B \times \frac{x}{3}}$$

$$+ \frac{10}{3} + \underline{M_B (4)} - \frac{160}{27} + \underline{\frac{M_B (16)}{27}} + 8 = 0$$

$$\frac{9}{9} \times M_A \left(\frac{4}{3}\right) + M_B \times \frac{180}{27} + \frac{126}{27} = 0$$

$$36 M_A + 180 M_B + 126 = 0$$

$$36 M_A + 180 M_B = -126 \quad \text{--- (2)}$$

$$9 \times (8 M_A - 20 M_B) = 0 \quad \text{--- (1)}$$

$$108 M_A = -126 \Rightarrow M_A = -1.0867 \text{ KNM.}$$

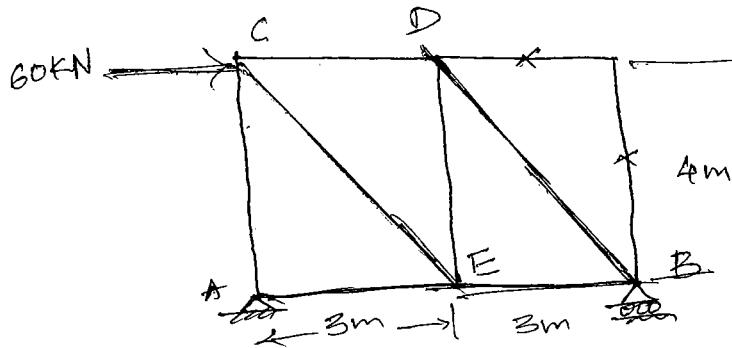
$$8 M_A = 20 M_B \Rightarrow M_B = -0.4333 \text{ KNM.}$$

$$M_C = 6 \text{ KNM} \checkmark$$

————— * —————

$M_A = -0.43 \text{ KNM} \checkmark$
 $M_B = +0.869 \text{ KNM} \checkmark$

~~Set 1~~ prob- compute the vertical deflection of Joint E by unit load Method? (12M)



$$\text{Ansatz: } I \times \Delta = \sum \frac{Q \cdot F \cdot L}{A \cdot E} : U = \int_0^L \frac{F_x^2 \, dm}{2 \cdot A \cdot E} \Rightarrow \frac{\partial U}{\partial F} = \Delta$$

$$W.D \Rightarrow I \times \Delta = \sum \frac{Q \cdot F \cdot L}{A \cdot E}$$

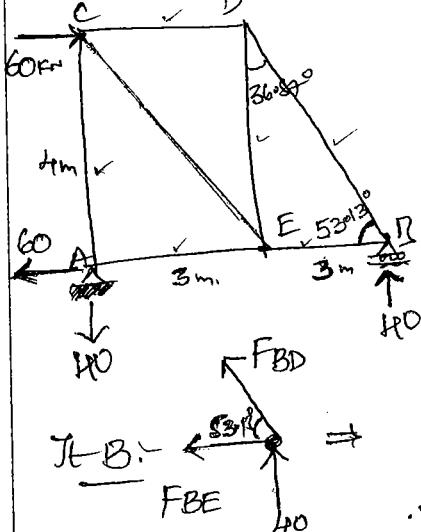
$$\Delta = \int_0^L \frac{\partial (F_n^2) \, dm}{2 \cdot A \cdot E}$$

$$\Delta = \int_0^L \frac{\cancel{\delta} F_n \left(\frac{\partial F_n}{\partial F} \right) \cdot dm}{\cancel{\delta} A \cdot E}$$

$$\therefore \Delta = \sum \frac{Q \cdot F \cdot L}{A \cdot E}$$

$\text{Q} \rightarrow$ Internal Membrane forces, due to " given loads
 $\text{F} \rightarrow$ " " " "

To find F_i

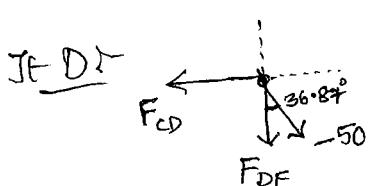


$$F_{AE} = 60 \text{ kN}$$

$$F_{AC} = 40 \text{ KN}$$

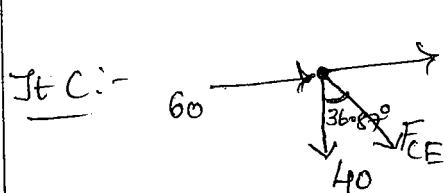
$$F_{BD} = \frac{-40}{\sin 53.13^\circ} = -50 \text{ kNm}$$

$$F_{BE} = 30 \text{ kNm}$$



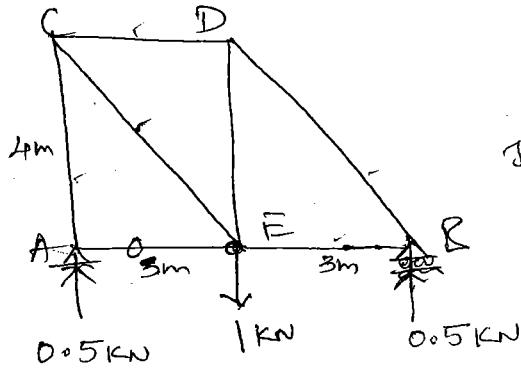
$$F_{DE} + (-50 \cos 36.87^\circ) = 0 \Rightarrow F_{DE} = 40 \text{ kNm}$$

$$F. \cos(36^\circ 87') = 0$$



$$F_{CE} = -50 \text{ kNm}$$

To find (Q)



JF A -

$$F_{AC} = -0.5 \text{ kN}$$

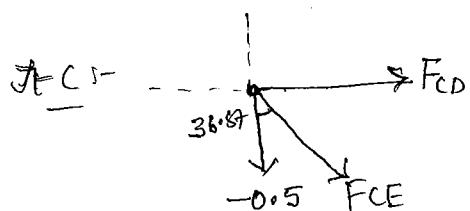
$$F_{AE} = 0$$

JF B - F_{BD}

$$F_{BD} = -0.625 \text{ kN}$$

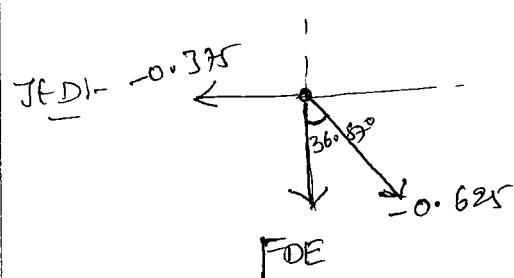
$$F_{BE} = 0.375 \text{ kN}$$

0.5



$$F_{CE} \text{ cos } 36.87^\circ - 0.5 = 0 \quad F_{CE} = 0.625 \text{ kN}$$

$$F_{CD} = -0.375 \text{ kN}$$



$$-0.625 \text{ cos } 36.87^\circ + F_{DE} = 0$$

$$F_{DE} = 0.5 \text{ kNm}$$

<u>Table 1</u> <u>Members</u>	<u>Length</u>	<u>Area</u>	<u>E</u>	<u>Q</u>	<u>$\frac{QFL}{AE}$</u>
AE	4	A	400	-0.5	-80/AE
AE	3	A	60	0	0
CD	3	A	-30	-0.375	33.75/AE
CE	5	A	-50	0.625	-156.25/AE
DE	4	A	40	0.5	80/AE
BD	5	A	-50	-0.625	156.25/AE
BE	3	A	30	0.375	33.75/AE

$$\sum = 67.5/AE$$

$$\therefore \Delta = \sum \frac{QFL}{AE} = \frac{67.5}{AE}$$

$$E = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^5 \times 10^{-3} / 10^{-6} = 2 \times 10^5 \times 10^3 = 2 \times 10^8 \text{ KN/m}^2$$

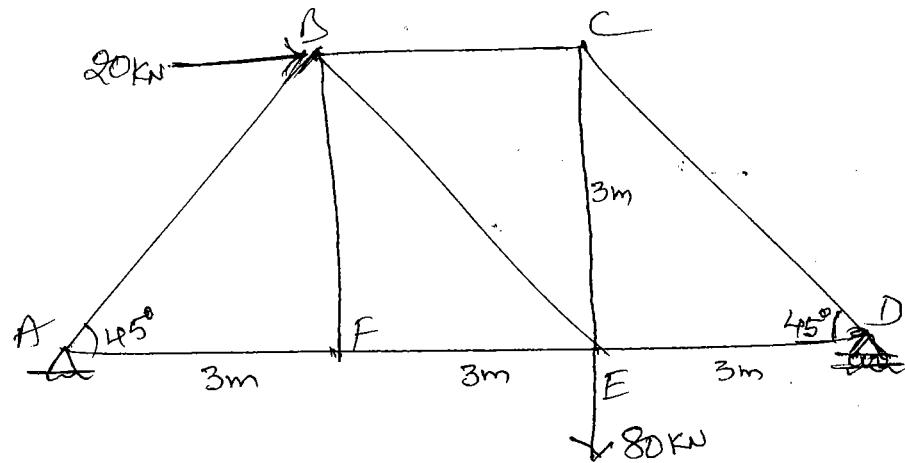
$$\text{Let, } A = 5000 \text{ mm}^2 = 5000 \times 10^{-6} \text{ m}^2 = 5 \times 10^{-3}$$

$$\Delta = \frac{67.5}{5 \times 10^{-3} \times 2 \times 10^8} = 67.5 \times 10^{-6} = 67.5 \times 10^{-3} \text{ mm}$$

$$\Delta = 0.0675 \text{ mm}$$

(7) - IV

~~Sol:~~ - Compute the vertical deflection of Joint E by unit load method? (8M)



Castigliano's First theorem:-

"The partial derivative of the total strain energy in any structure with respect to applied force/moment gives the defl. or rotation resp. at the point of application of the force/moment in the direction of applied force/moment."

MOVING LOADS AND INFLUENCE LINES

Rolling

Moving Loads! - max S.F and BM at given section, and also max SF & BM due to single point load, UDL longer than span and shorter than span. Two point loads with fixed distance, Several point loads - equivalent UDL - Focal length.

Intro - Eg: Vehicle movement across the bridge girder, (EOT) Electrically Operated Travelling overhead crane across Gantry girders in an industry, Train on a bridge across a river, etc.

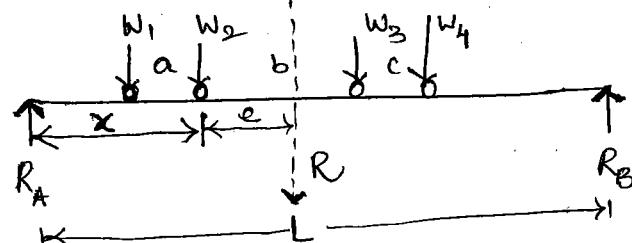
How to position live load to maximize the value of S.F, BM and support reaction in the beam.

Max Shear Force and Bending moment at a given section

In case of moving loads, BM and SF at a section of a beam change as the loads roll from one position to the other.

- To determine the load positions for max. BM or S.F for a given section of a girder and to compute its value.
- To determine the load positions so as to cause absolute maximum BM or S.F anywhere on the girder.

Let the position of load w_2 when bending moment under this load is maximum.



$$\therefore R_A = \frac{R(L-x-e)}{L}$$

BM under load w_2 is,

$$\text{for max. } M_2, \frac{dM_2}{dx} = 0 \Rightarrow \frac{R(L-e-2x)}{L} = 0$$

$$L-e-2x=0$$

$$L-e=2x \Rightarrow x = \frac{L-e}{2}$$

$$M_2 = R_A x - w_2 a$$

$$= \frac{R(L-x-e)x}{L} - w_2 a$$

- Bending moment under a perpendicular load is maximum when the centre of the beam is midway between that load and the resultant of all loads there on the span.
- max. SF occurs at supports and is equal to the max. reaction. maximum Reaction is the reaction to which the resultant load is nearest.

Absolute max. SF and BM :- Conditions

- Absolute maximum Shear force V_{abs} occurs at one of the two end supports and is numerically equal to the maximum support reactions.
- By placing each load alternatively over a support and calculating the reactions, the absolute max. S.F V_{abs} can be calculated. The critical position usually occurs when the largest load is on the support with as many remaining loads as possible on the span. It is to be borne in mind that the load can move in either direction.
- Due to a system of concentrated loads moving across the span of a simple beam, the absolute maximum moment M_{abs} always occurs under a large load located near the resultant of all the loads on the beam.
- Critical load that causes a max. moment must be obtained by the trial and error procedure. The abs. max. moment can be found when the distance of the critical load and the resultant of the loads on the beam is bisected by the centre line of the beam.

Single point load:-

→ If load is placed at a distance γ from A,

$$\text{Max. } \leftarrow \text{S.F} = R_B = -\frac{W\gamma}{L} \quad (0 \leq \gamma \leq x)$$

$$\text{Max. } \leftarrow \text{S.F} = -\frac{Wx}{L} \quad \therefore \text{when } \gamma = x \\ V_{x,\max}$$

$$x=0, V_{x,\max} = 0$$

$$x=L, V_{x,\max} = -W = V_{\text{abs}}$$

∴ Absolute maximum \leftarrow S.F occurs at right-hand support,

its value is $-W$

$$\Rightarrow V_x = R_A = \frac{(L-\gamma)W}{L}$$

$$\text{when } \gamma = x, V_{x,\max} = \frac{W(L-x)}{L}$$

$$\text{when, } x=0, V_{\text{abs}} = W$$

∴ Absolute max the S.F occurs at left Support with a Value "W".

$$\Rightarrow \text{Max. } BM: \text{ when } \gamma \text{ is b/w A & X, } BM_x \text{ is equal to } R_B \cdot (L-x) = \frac{W\gamma(L-x)}{L}$$

$$\text{when } \gamma = x, M_x = \frac{Wx(L-x)}{L}; \text{ when } \gamma = 0, M_x = 0$$

when Load is between $X \& B$,

$$M_x = R_A x = \frac{W(L-\gamma)x}{L}$$

$$M_{x,\max} = \frac{W(L-x)x}{L}$$

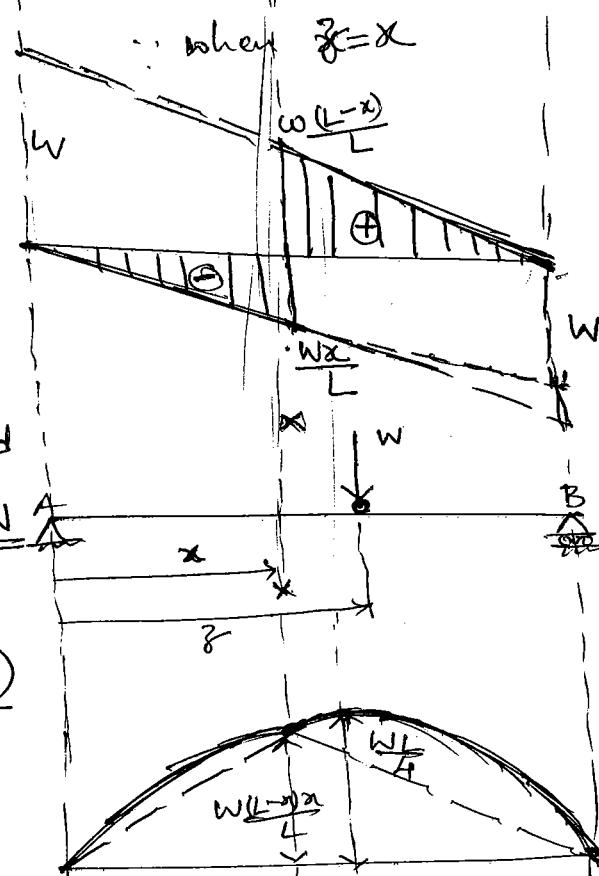
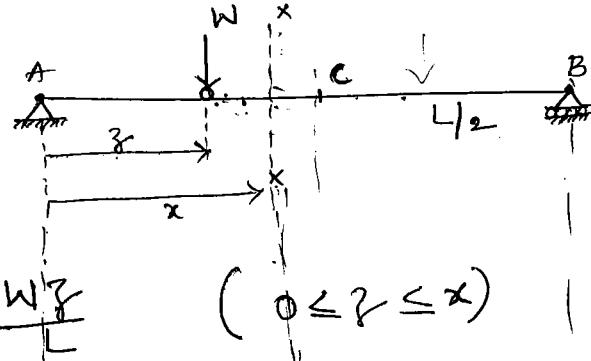
when $\gamma = L, M_x = 0$

$$M_{x,\min} = \frac{Wx^2}{2} - \frac{Wx^2}{L} = \frac{Wx^2}{L}$$

$$\frac{dM}{dx} = 0 \Rightarrow \frac{W-2Wx}{L} = 0$$

$$x = \frac{L}{2}$$

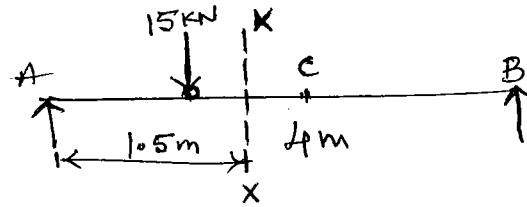
$$\therefore \text{Abs. Max. } BM = \frac{WL}{4}$$



Prob 1 - Det more the ues -
1.5m in a simple beam of Span 4m when a Concentrated load of 15 kN acts across the beam. Also calculate the Absolute max and the shear and BMs.

Sols-

$$V_{x, \text{max}} = \frac{-15 \times 1.5}{4} \\ = -5.625 \text{ kN}$$



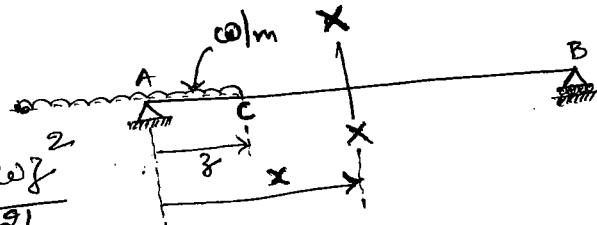
$$V_{\text{max}^+} = \frac{15(4-1.5)}{4} = 9.375 \text{ kN}$$

$$M_{\text{max}} = \frac{15 \times 1.5 \times 2.5}{4} = 14.0625 \text{ kNm}$$

$$V_{\text{abs}} = \pm 15 \text{ kN}; M_{\text{abs}} = \frac{15 \times 4}{4} = 15 \text{ kNm}$$

UDL Longer than Span:-

$$V_x = -R_B = -\frac{\omega \frac{x^2}{2}}{L}$$



when, head of load reaches the section. $x=L$

$$V_{\text{max}} = -\frac{\omega x^2}{2L}; \quad \begin{cases} \text{when } x=0, V=0 \\ x=L, V = -\frac{\omega L^2}{2} \end{cases} \quad \text{parabolic}$$

$$V_x = R_A = \frac{\omega(L-x)^2}{2L} \Rightarrow V_{x, \text{max}^+} = \frac{\omega(L-x)^2}{2L}$$

when $x=0, V = \frac{\omega L^2}{2}$; parabolic
 $x=L, V = 0$

Max. BM :- $M_x = R_B x(L-x)$
 $= \frac{\omega x^2}{2L}(L-x)$

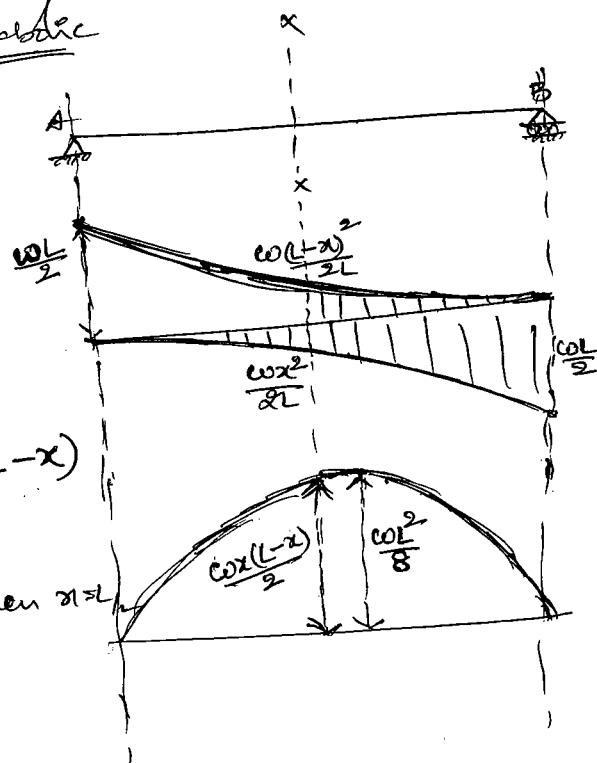
$$M_x = \frac{\omega x^2(L-x)}{2L}$$

$$M_{x, \text{max}} = \frac{\omega L x}{2} - \frac{\omega x^2}{2} = \frac{\omega x}{2}(L-x)$$

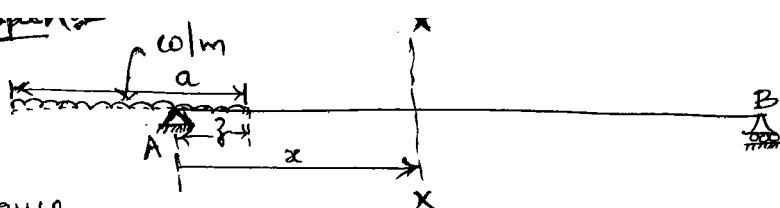
Load occupies the whole span.

$$M_{\text{max}} = \frac{\omega x}{2}(L-x) \quad \text{when } x=L$$

$$M_{\text{max}} = \frac{\omega L^2}{8}$$



UDL over main span

Negative S.F :-

Case(I) - When the distance of the section is shorter than the length of load ($x < a$)

$$SF \text{ at } x \text{ is } V_x = -R_B = -\frac{\omega z^2}{L} = -\frac{\omega z^2}{2L}$$

$$\text{at } z=x, V_{max} = -R_B = -\frac{\omega x^2}{2L} \quad x=0, V_x=0 \quad x=a, V_{max} = -\frac{\omega a^2}{2L}$$

Case(II) :- $x > a$; when head of load reach 'x'.

$$V_{max} = -R_B = -\frac{\omega a}{L} \cdot \left(x - \frac{a}{2}\right)$$

$$x=a/2 \quad V_{max}=0$$

$$x=a \quad V_{max} = -\frac{\omega a^2}{2L}$$

$$x = \left[\frac{a}{2} + \frac{a}{2}\right] \Rightarrow V_{max} = -\frac{\omega a}{L} \left(\frac{L+a}{2} - \frac{a}{2}\right)$$

$$x=L \quad V_{max} = \frac{-\omega a(L-a)}{L}$$

$$= -\underline{\underline{\omega a}},$$

$$x=L+\frac{a}{2}$$

- * Max. negative S.F at a section occurs when head of the load is on the section. (Load to the left of the section)

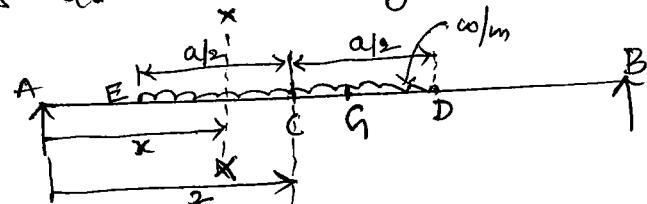
- * Max. positive S.F at a section occurs when the tail of the load is on the section

- * Absolute max. negative S.F occurs at support B when head of the load is at B.

- * Absolute max. positive S.F occurs at support A when tail of the load is at A.

BM :- when load is in AX portion, BM at section 'x' is expressed as $M_x = R_B (L-x)$

to get maximum BM at the section 'x' the load has to be arranged in such a manner that CG of the load is at distance $\frac{x}{2}$ from 'A'.



$$R_B = \omega a (z/L)$$

$$XD = (z - x + a/2)$$

$$M_x = R_B (L-x) - \frac{\omega (XD)^2}{2}$$

$$M_x = \frac{w}{L} x (L - x) - \frac{w}{2} (x - \frac{a}{2})$$

for M_x to be maximum, $\frac{dM_x}{dx} = 0$

$$\Rightarrow \frac{w}{L} (L-x) - \frac{w}{2} (x - x + \frac{a}{2}) \times 2 = 0$$

$$\frac{a}{L} (L-x) = (x - x + \frac{a}{2})$$

$$\frac{ED}{AB} \times XB = XD \quad \text{or} \quad \frac{XB}{XD} = \frac{AB}{ED} = \frac{AB - xB}{ED - xD}$$

$$\therefore \frac{ED}{XD} = \frac{AB}{XB}$$

* BM at a section will be maximum when the position of load is such that, the section divides the span and the load in the same ratio.

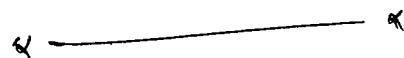
$$x = \frac{a(L-x)}{L} + x - \frac{a}{2} = \frac{a}{2} + x - \frac{ax}{L}$$

$$M_{max} = \frac{w}{L} a (L-x) \left(\frac{a}{2} + x - \frac{ax}{L} \right) = \frac{w}{2} \left(\frac{a(L-x)}{L} \right)^2$$

$$= \frac{wax}{L} (L-x) \left(1 - \frac{a}{2L} \right)$$

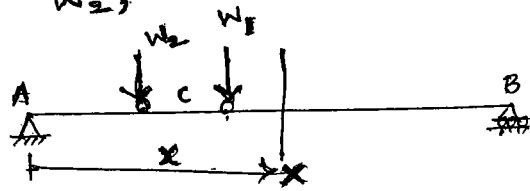
Absolute max BM occurs at centre, $x = \frac{L}{2}$

$$M_{abs} = \frac{w}{4} a (L - \frac{a}{2})$$



Assuming w_1 is lighter than w_2 ,

SF:- case(i) Both loads to the left of Section 'x'



case(ii) w_1 to the right of 'x' and w_2 to the left of it.

case(iii) Both loads to the right of section

Case(i):- Both loads to the left of 'x'; $V_x = R_B$

(a) when $x < c$, only w_1 will be on the beam, w_2 outside the span with w_1 on 'x'.

$$\therefore V_{0/x} = -R_B = -\frac{w_1 x}{L}$$

(b) when $x > c$, both loads will be on the beam with w_1 at 'x'.

$$V_{0/x} = -R_B = -\frac{w_1 x + w_2 (x-c)}{L}$$

case(ii) w_1 to the right and w_2 to the left of 'x'

$$V_x = -R_B + w_1$$

(c) $(L-x) > c$, both w_1 and w_2 will be on the beam w_2 on the 'x'.

$$V_{0/x} = -R_B + w_1 = -\frac{w_2 x + w_1 (x+c)}{L} + w_1$$

(d) $(L-x) < c$, w_1 will be outside the girder, w_2 at 'x'

$$V_{0/x} = -R_B = -\frac{w_2 x}{L}$$

One of the above four expressions will give

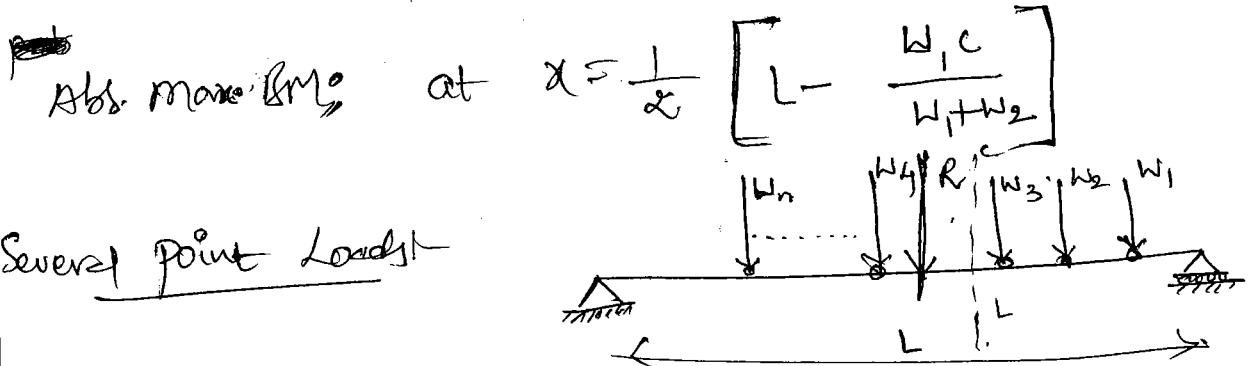
max. -ve SF.

(i) $x=0$ to $x=c$; (ii) $x=c$ to $x=L-c$; (iii) $x=(L-c)$ to L

* max. -ve SF occurs only when both the loads are to the left of the section with w_1 approaching it.

* min. the SF occurs only when both the loads are to the right of the section.

~~more~~ Avg load on the left of section
is equal to the avg load on right of section



Several Point Loads

max. S.F. - Thrust and Error.

max BM under a given load when centre of beam is midway b/w the load and resultant of all loads!

Max BM over a given section avg load on left =
" " Right.

Abs. Max BM Abs. max BM occurs under one of the loads. It occurs under heavier load nearer the mid span.

Equivalent UDL: Single point load

$$M_{\text{max}} = \frac{w_{\text{eq}}}{L} (L - x)$$

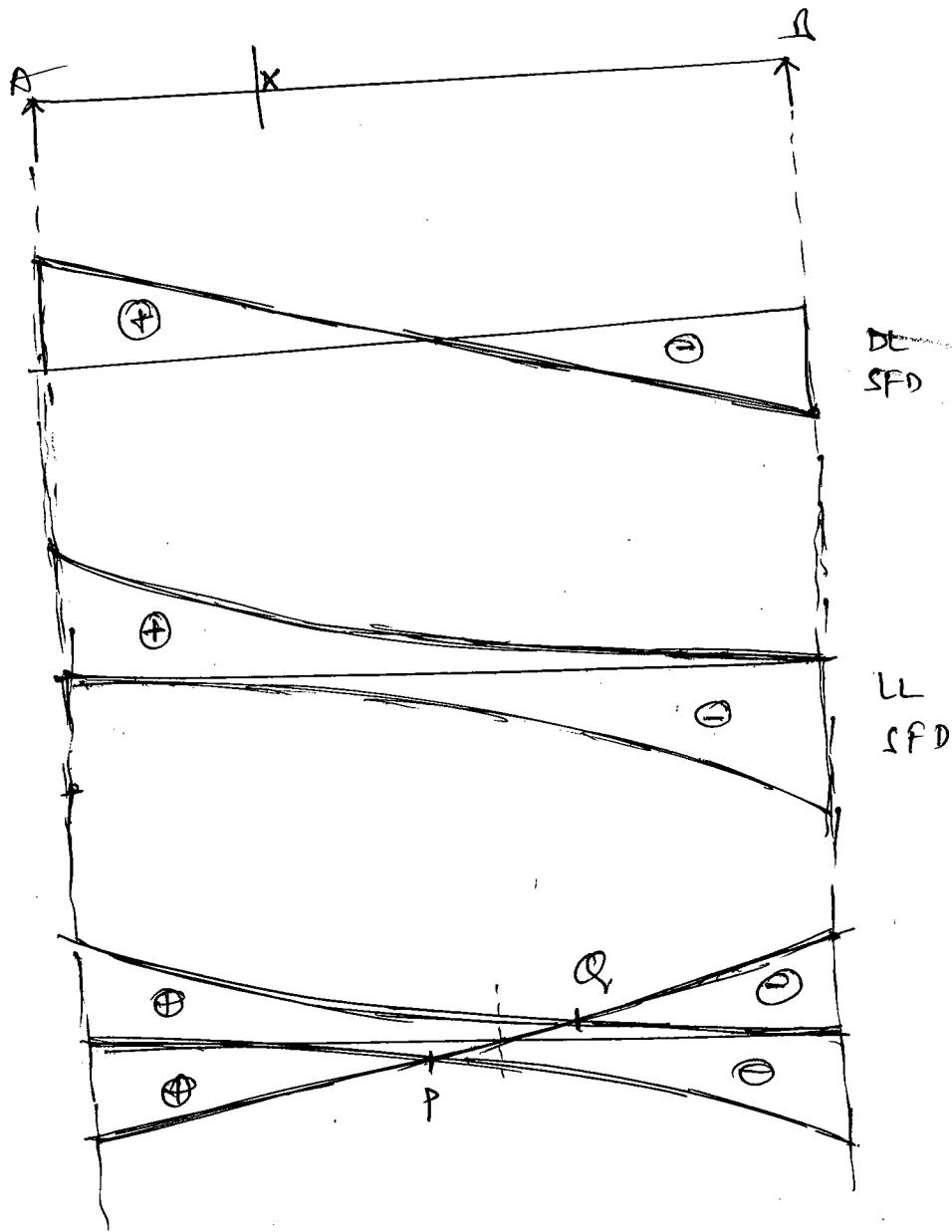
$$M_s = \frac{w_{\text{eq}} L}{2} - \frac{w_{\text{eq}} x^2}{2} = \frac{w_{\text{eq}}}{2} (L - x)$$

$$\therefore w_{\text{eq}} = \frac{2M}{L}$$

UDL shorter than span $M_{\text{max}} = \frac{w_{\text{eq}}}{4} (L - x)^2$

$$M_s = \frac{w_{\text{eq}} L^2}{8} \Rightarrow w_{\text{eq}} = \frac{8M}{L^2} (L - x)$$

focal lengths: The portion over which shear force changes sign due to dead load and live load (moving) is called the focal length.



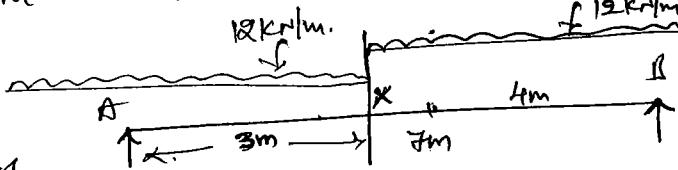
PQ is focal length of girder A-B.

To find focal length, $V_{DL} + V_{LL} = 0$ gives
 x which is half of the ~~length~~ outside focal length
 due to symmetry = $2x$.
 \therefore focal length = $L - 2x$

~~prob~~ Det max positive and negative shear force and bending moment at a section 1.5m from a simple beam of span 4m when a concentrated load of 15 kN acts across the beam. Also, calculate the absolute negative and positive shears and bending moments?



prob A UDL of intensity 12 kN/m and length more than 4m moves across a girder of span of 7m. Find the maximum positive and negative shear force at a section 3m from left support as well as its absolute value. Similarly, determine the maximum BM at the same section and the absolute value?



Ans max +ve S.F occurs

when head of the UDL is on X,

$$V_{\text{max}} = \frac{12 \times 3 \times 1.5}{7} = -7.14 \text{ kN}$$

max. the S.F occurs

when the tail of the UDL is on X',

$$V_{\text{max}} = \frac{12 \times 4 \times 2}{7} = 13.714 \text{ kN}$$

Absolute maximum S.F occurs when the Head of UDL is on B (or) the tail is on A.

$$\therefore V_{\text{abs}} = \pm \frac{12 \times 7}{2} = 42 \text{ kN}$$

Max BM at X:- when load occupies the whole span,

$$= \frac{1}{8} \times 3 \times \left(\frac{12 \times 3 \times 4}{7} \right) \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} \times \left(\frac{12 \times 3 \times 4}{7} \right) \times \frac{1}{2}$$

$$\text{Ans BM} = \frac{12 \times 3 \times 4}{8} \times \frac{1}{2} [3.5] = 72 \text{ KNM}$$

Influence Lines - IL for SF & BM - load position for max SF at a section - load position for max BM at a section. Single Point Load \Rightarrow UDL Longer than Span. UDL Shorter than Span - IL for forces in members of pratt and warren trusses!

(6) - VII

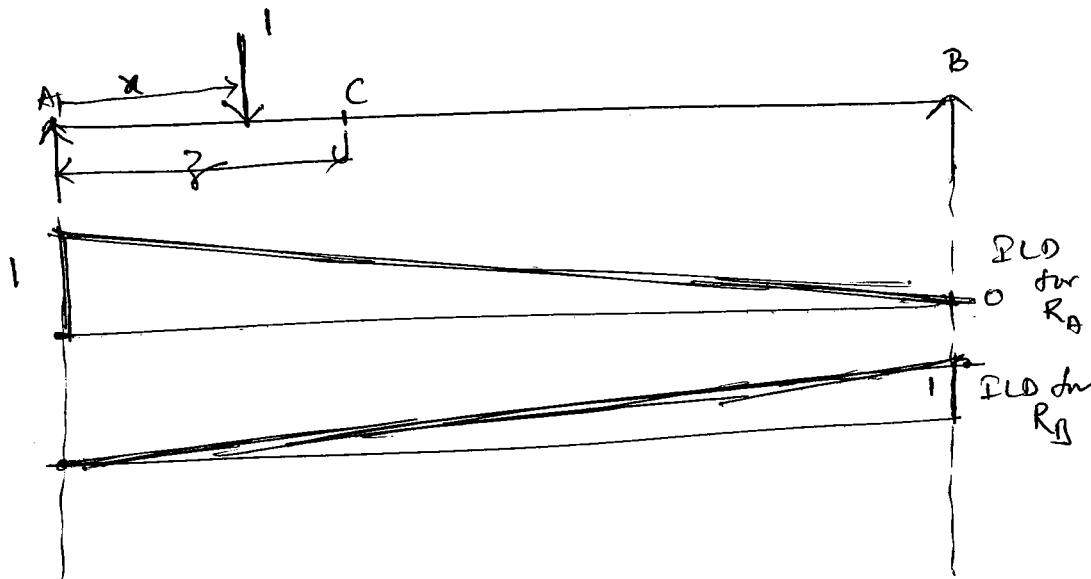
Introduction Loads acting on a beam may be broadly classified as dead loads and live loads.

- * Loads which do not change their position during the life of the beam are DL.
- * Loads which can change their position during the life of the beam are L.L

ILD - ILD for a Stress resultant is the one in which ordinate represent the value of the stress resultant for the position of unit load at the corresponding abscissa

ILDs for SSB :-

(i) ILD for reaction R_A :-



$$R_A = \frac{1-(L-x)}{L} = 1 - \frac{x}{L}$$

$x=0 \quad R_A = 1$
 $x=L \quad R_A = 0$

for R_B :

$$R_B = \frac{x}{L}; \quad x=0 \quad R_B = 0$$

$x=L \quad R_B = 1$

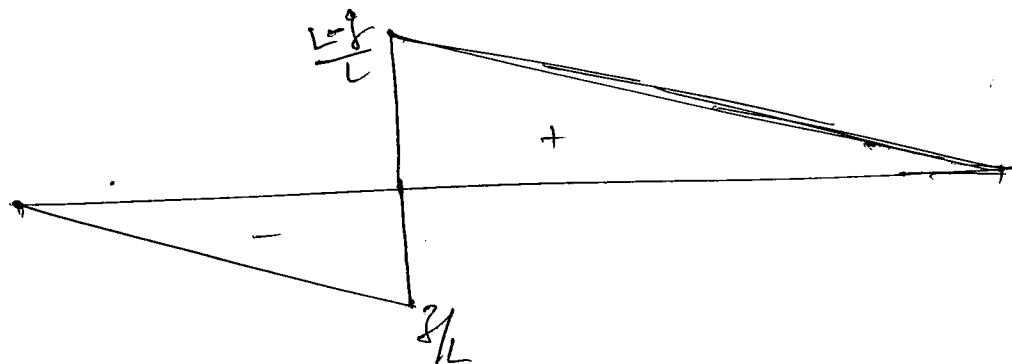
LD für SF_C :

a) when $x < \gamma$, $SF_C = \frac{-x}{L}$

$x=0 \quad SF = 0$
 $x=\gamma \quad SF = -\frac{\gamma}{L}$

b) " $x > \gamma$, $SF_C = R_A = \frac{L-x}{L}$

$x=\gamma \quad SF_C = 0$



BM_C : when $x < \gamma$: $M_C = R_B \cdot (L-\gamma)$

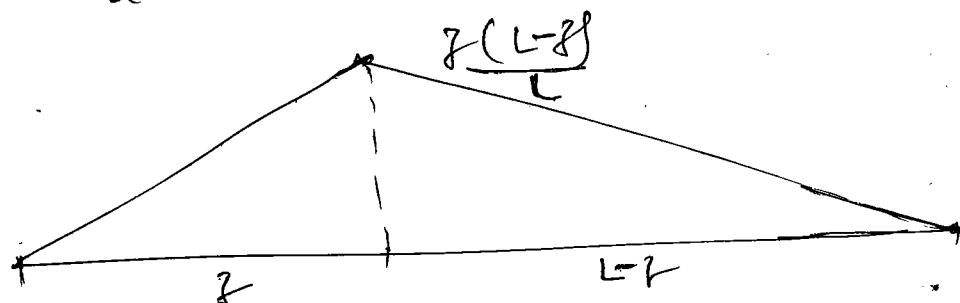
$$= \frac{x}{L} (L-\gamma)$$

$x=0 \quad M_C = 0$
 $x=\gamma \quad M_C = \frac{\gamma(L-\gamma)}{L}$

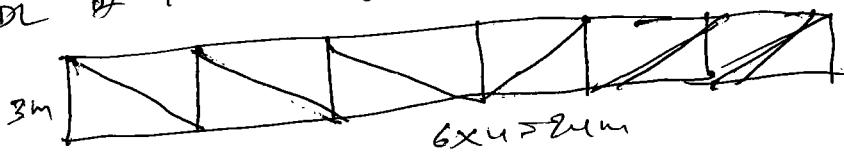
$x > \gamma$: $M_C = R_A \gamma = \frac{(L-x)}{L} \cdot \gamma$

$x=\gamma \quad M_C = \frac{(L-\gamma)}{L} \cdot \gamma$

$x=L \quad M_C = 0$



- 1) Draw ILD for the bending moment in a section X for a simply supported beam? (3m)
- 2) Draw ILD for the reaction at A for a SSB AB? (4m)
- 3) Two point loads of 600N and 300N spaced at 4m apart cross a girder of 10m span from left to right, with smaller load lesser draw SF and BM diagrams. Find the position and amount of absolute max. BM? (16m)
- 4) A uniform load of 2000N/m, 5m long crosses a girder of 20m span from left to right. Calc. the max. S.F. and BM at a section 8m from left support? (16m)
- 5) Draw ILD for the reaction at B S.F. and BM for a simply supported beam AB? (3m)
- Section X for a reaction at B for SSB? (1m)
- 6) draw ILD for forces in the members of a Warren truss? (6m)
- 7) what are the positions of a single load for maximum BM at section and absolute BM in the span? (4m)
- 8) A SSB of span 8m is loaded with three point loads of 5kN, 10kN and 15kN, at a distance 2m, 4m and 6m resp from right end. It also carries UDL of 10kN/m throughout the span. Find position and magnitude of max. deflection and calculate max S.F? (16m)
- 9) Define ILD and draw ILD? (4m)
- 10) Define ILD and draw ILD? (4m)
- 11) A UDL of 40kN/m and of length 3m transverse across the span of SSB of length 16m. Compute max BM at 4m from left support and absolute BM? (16m)
- 12) Find max force in the members shown if UDL of 10kN/m longer than span crosses abridge when



Single point Loads

- (i) Max +ve S.F occurs when load is just to the left of section 'C' = $\frac{Wz}{L}$
 - (ii) max -ve S.F occurs when load is just to the right of section 'C' = $W \left(\frac{L-z}{L} \right)$
 - (iii) max BM will occur when the load is on the section itself = $W z \left(L-z \right)$
- Abl. max S.F = Wz one at B. and the at A.
 " " BM = $\frac{WL}{4}$ for $z = \frac{L}{2}$ "

UDL Longer than Span :-

$$\text{Max. +ve S.F.} = \frac{\omega z^2}{2L} \quad (\text{load } \cancel{\text{to sec C}})$$

$$\text{the S.F.} = \frac{\omega (L-z)^2}{2L} \quad (\text{load } \cancel{\text{to sec B}})$$

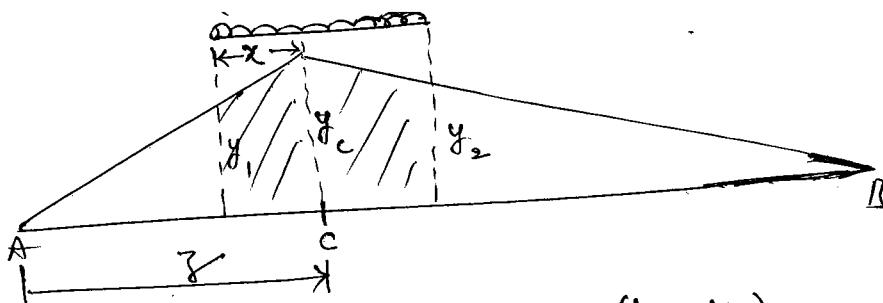
$$\text{max BM, UDL on whole span,} = \frac{\omega z(L-z)}{2}$$

$$\text{Abl. max S.F.} = \frac{\omega L}{2}$$

$$\text{BM} = \frac{\omega L^2}{8}$$

UDL smaller than Span

- (i) for max +ve S.F the tail of UDL should reach the section.
- (ii) for max -ve the head of the UDL reach the section



$$M_C = \omega \times x \left(\frac{y_1 + y_c}{2} \right) + \omega (d-x) \left(\frac{y_c + y_2}{2} \right)$$

for M_C to be maximum, $\frac{dM_C}{dx} = 0$,

$$\omega \left(\frac{y_1 + y_c}{2} \right) - \omega \left(\frac{y_c + y_2}{2} \right) = 0$$

$$\boxed{y_1 = y_2}$$

$$\therefore \frac{(z-x)}{z} \times y_c = \frac{(L-z) - (d-x)}{(L-z)} \times y_c$$

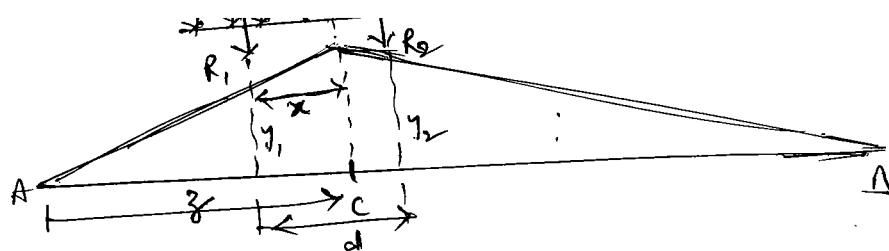
$$Lx = dz$$

$$\boxed{\frac{x}{d} = \frac{z}{L}}$$

- * B.M is maximum at a section when the load is so placed that the section divides the load in the same ratio as it divides the span.
- * For absolute maximum moment c.g of the load will be at the mid-span

Train of point loads-

- * for max-u.s.f at a section, most of the loads are to be to the left of section
- * for maximum t.s.f at a section, most of the loads are to be to the right of the section



$$M_c = R_1 y_1 + R_2 y_2 = R_1 \frac{(l-x)}{z} \times y_c + R_2 \frac{(c-x)}{z} \times y_c$$

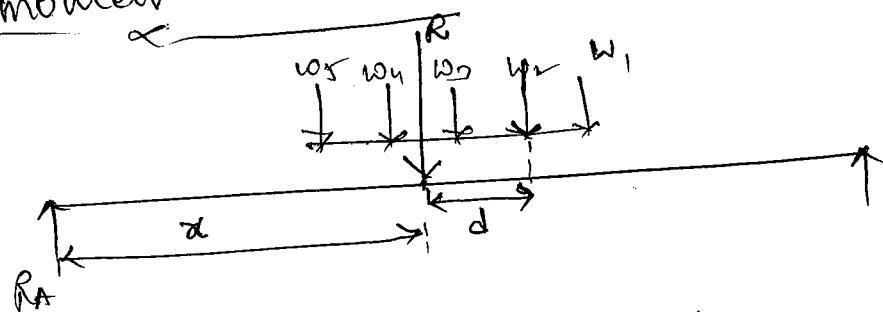
for M_c to be maximum $\frac{dM_c}{dx} \leq 0$

$$\boxed{\frac{R_1}{z} = \frac{R_2}{l-z}}$$

"Avg load on left portion of the beam is same as the Avg load on right side portion."

Absolute Max S.F.:- happens at the supports.

Maximum moment under a load-



$$R_A = \frac{R(L-x)}{L}; M_{w_2} = R_A(x+d) - R_D d$$

$$= \frac{R(L-x)(x+d)}{L} - Rd$$

$$\frac{dM}{dx} \leq 0 \Rightarrow \frac{R}{L}(L-x-d) \leq 0$$

$$x = \frac{L}{2} - \frac{d}{2}$$

$$\text{Distance of } w_2 \text{ from } A = x+d = \frac{L}{2} + \frac{d}{2}$$

"For moment to be max under a load, the load and resultant should be equidistant from the mid-span".

Absolute maximum BM: Absolute maximum moment occur under one of the loads when the resultant of all the loads and the load under consideration are equidistant from the centre of the beam.

Prob Two point loads of 6KN and 3KN Spaced at 4m apart cross a girder of 10m span from left to right, with smaller load leading. Draw SF and BM diagrams. Find the position and amount of abs max. BM & (16M)

Soln $W_1 = 3\text{KN}$ $W_2 = 6\text{KN}$, $c = 4\text{m}$ $L = 10\text{m}$

$$\text{Condition, } c < \frac{W_1 L}{W_1 + W_2} \Rightarrow \frac{(3 \times 10)}{(3+6)} = \frac{30}{9} = 3.333$$

Hence $c > \frac{W_1 L}{W_1 + W_2}$

$$V_{ax} = -\frac{W_2 x + W_1(x-c)}{L} = -\frac{6x + 3(10-x)}{10} = -\frac{3x + 30}{10} \quad \text{eq. 18.26}$$

$$V_{ax} = -\frac{W_2 x + W_1(x+c)}{L} + W_1 = -\frac{6x + 3(4+x)}{10} + 3 = -\frac{9x + 12}{10} + 3$$

$$V_{ax} = -\frac{W_2 x}{L} = -\frac{6x}{10} = -\frac{3x}{5} \quad \text{Eq. 22}$$

$$(L-x) \neq c \Rightarrow x \neq 4 \Rightarrow x \neq 4$$

$$x=0, V_{ax} = 0$$

$$x=4\text{m}, V_{ax} = -\frac{6 \times 4}{10} = -2.4\text{KN}$$

$$(L-x) \neq c \Rightarrow x=4\text{m}, V_{ax} = -\frac{6x + 3(4+x)}{10} + 3 = -\frac{10x + 12}{10} + 3 = -1.08\text{KN}$$

$$x=5\text{m}, V_{ax} = -\frac{6x + 3(4+5)}{10} + 3 = -\frac{15x + 27}{10} + 3 = -2.7$$

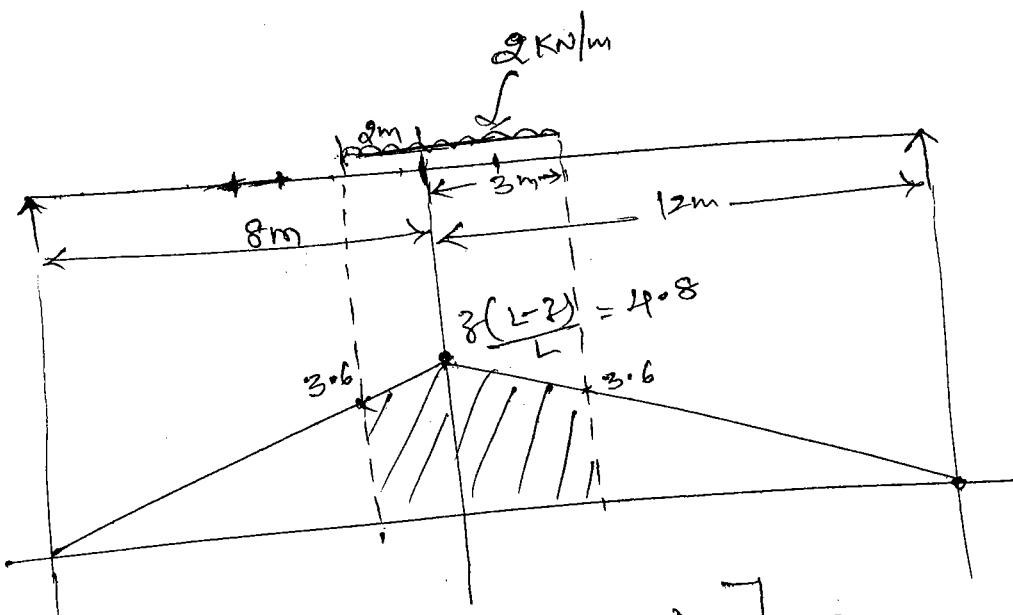
$$x=10, V_{ax} = -\frac{6x + 3(4+10)}{10} + 3 = -7.2$$

prob1- A uniform load of 2000 N/m, 5m long crosses a girder of 20 m span from left to right. Cal. max S.F and BM at a section 8m from left support?

Sol:-

$$\frac{x}{d} = \frac{8}{L}$$

$$\frac{x}{5} = \frac{8}{20} \Rightarrow x = \frac{8 \times 5}{20} = 2 \text{ m}$$



$$\begin{aligned} \text{max BM at } 8\text{m} &= \left[\left(\frac{3.6 + 4.8}{2} \right) \times 2 + \left(\frac{3.6 + 4.8}{2} \right) \times 3 \right] \times 2 \\ &= \left(\frac{3.6 + 4.8}{2} \right) \times 5 \times 2 \\ &= 48 \text{ KNm} \end{aligned}$$

for max S.F:-

$$= \left(\frac{0.6 + 0.35}{2} \right) \times 5 \times 2$$

$$= 4.75 \text{ KN}$$

